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DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Md. 20084

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A MATHEMATICAL INTRODUCTION TO SHIP MANEUVERABILITY

THE SECOND DAVID W. TAYLOR LECTURES

SEPTEMBER 1973

by

Roger Brard

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October 1975

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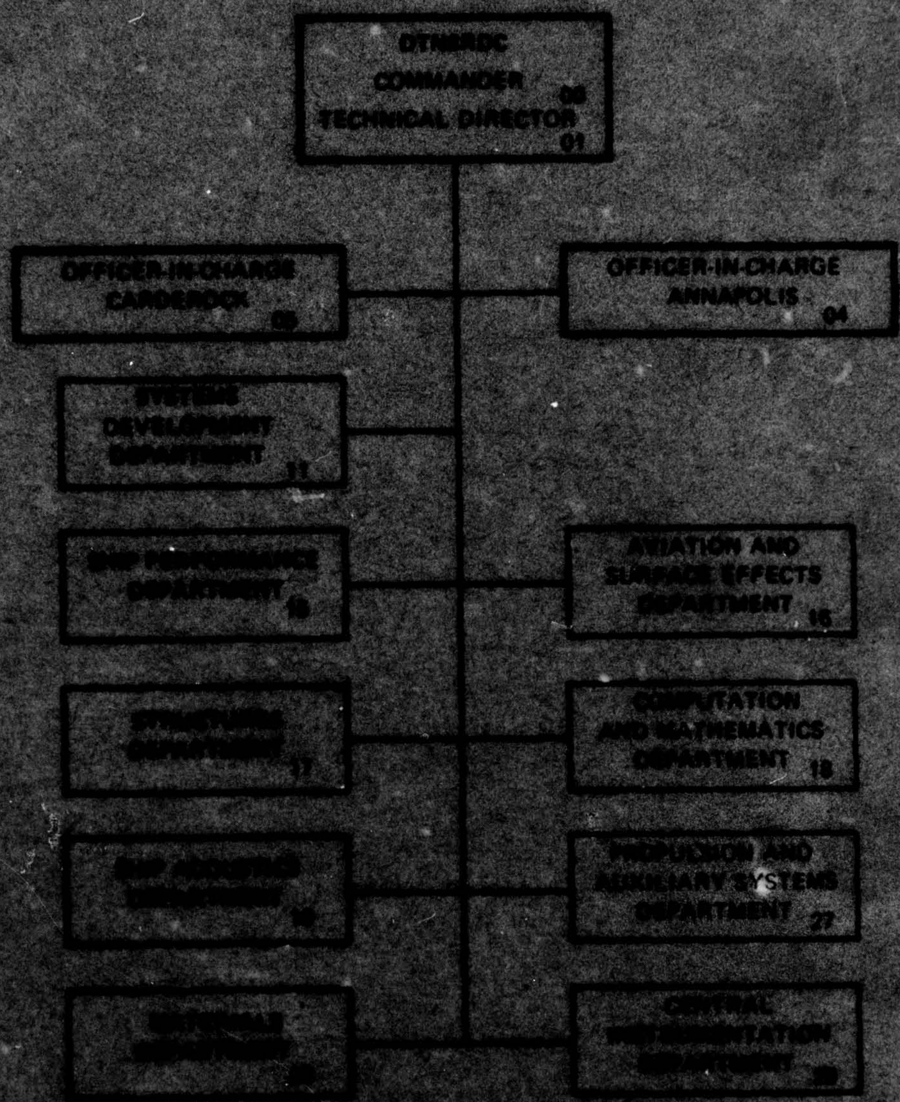

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PREFACE

The David W. Taylor Lectures were initiated as a living memorial to our founder in recognition of his many contributions to the science of naval architecture and naval hydromechanics. We invite eminent scientists in fields closely related to the Center's work to spend a few weeks with us, to consult with and advise our scientific staff, and to give lectures on subjects of current interest.

Admiral Taylor was a member of that very select group, a naval officer who was also a world renowned scientist. His research retains its significance today. He was a pioneer in the use of hydrodynamic theory for solving naval problems, in the use of mathematics for hull form development, and in his "global" treatment of ship resistance. As a naval officer, he founded and directed the Experimental Model Basin. He personally influenced the design of virtually all ships of the U. S. Navy for many years. Thus, he was more than a scientist, he was a manager and working engineer who helped shape a great fleet.

This year our lecturer is Vice Admiral Roger Brard of the French Navy. As the director of the Bassin d'Essais des Carènes, in Paris, for many years, he contributed to the design of many ships. And his scientific work in naval hydrodynamics has been rewarded by the highest recognition our profession can offer. Thus, in every way, his career is a modern parallel to that of Admiral Taylor. We are most honored that he has agreed to be a David W. Taylor Lecturer.

I present Roger Brard, Vice Admiral of the French Navy, former President of the Académie des Sciences of France, naval architect, mathematician, hydrodynamicist, and friend.

INTRODUCTION

May I first thank Dr. Cummins for his kind words of welcome. Please believe that I consider them much too laudatory. It is a great honor and privilege for me to deliver this Second Series of David Taylor Lectures. But it is also a dangerous task, in that it is nearly impossible to give lectures worthy of such a great man.

Admiral Taylor combined a genial intuition, a scientific spirit, and an astonishing sense of the physical possibilities. His achievements deserve admiration. Despite the great strides made in naval architecture since his day, it is still difficult to design ship forms superior to those of the Taylor Standard Series.

The subject selected for these lectures concerns ship maneuverability. This is a field where theory occupies a much less important place than empiricism because of the complexity of the phenomena involved in the maneuvering of a ship. No complete theory yet exists, and one can presume that this will be so for many years.

Nevertheless, it can be fruitful to gather some facts revealed by experiments and to connect them with relatively simple theoretical concepts. This does not lead to what a theory should be but rather to an introduction to such a theory. This is the purpose of the present lectures.

Two facts seem to be of great importance:

One is that a turning ship experiences a system of hydrodynamic forces, one component of which is orthogonal to the velocity of her center of gravity. This component must be considered as a lift. But, in the classical theory of lifting surfaces, a lift appears only if free vortices are shed. It follows that the vortex theory should be the basic tool of the maneuverability theory.

The second important fact is that the inertial forces of the fluid may depend not only on the motions of the body itself but also on the existing motion of the fluid when the maneuver occurs. This means that the maneuverability theory has to take into account not only the classical "added masses" but also "apparent added masses."

It follows from the above remarks that in order to obtain an expression for the classical added masses in terms of bound vortex distributions, we must first examine how the vortex theory applies when no free vortices are shed by the body. Then we have to generalize the classical lifting surface theory to the case of bodies with a finite displacement. This must be done even when there is no singular line on the hull which could play the role of the trailing edge of a wing.

Now, let me outline briefly the contents of these lectures. They are divided into seven chapters.

Chapter 1 comments in a concise manner on the concept (which will be used throughout the lectures) of an "almost" inviscid fluid, on the state of the art, and also on the optimizations of the trajectories of a body moving in water even though the control of such a body is beyond the scope of these lectures.

Chapter 2 summarizes the classical theory of the motion of a body moving through an inviscid fluid. It recalls the concept of the added masses according to which the inertial forces of the fluid are derived from its finite kinetic energy.

Chapter 3 treats the same problem but with the help of vortex theory. The moving body is replaced by a vortex distribution kinematically equivalent to the body inside the domain D_e exterior to the hull and actually occupied by the fluid. The domain D_i interior to the hull is occupied by a fictitious fluid mass which is at rest with respect to the hull. By stating that the set of fluid particles belonging to the vortex distribution is in dynamical equilibrium, we obtain a generalized form of the Kutta-Joukowski theorem concerning the hydrodynamic forces exerted on a bound vortex sheet. Furthermore, we can now state the expression for the hydrodynamic pressure on the hull. This may help to throw light on the concept of added masses and apparent added masses.

Chapter 4 begins with a review of the theory of thin wings, continues with the theory of wings of finite thickness, and then deals

with the case of moving bodies which have no real trailing edge. In our opinion, the lifting forces exerted on such bodies are due to a shedding of free vortices; this phenomenon occurs because of the intersection of trajectories of fluid points coming from the suction side with those coming from the pressure side. Two examples are selected. One could be that of a submarine which moves in the vertical plane and the other that of a double model moving in the horizontal plane. The problem would be underdetermined if the continuity of the pressure through the free vortex sheets did not entail the continuity through the shedding line of the pressure on the hull. Here, we encounter a difficulty because the position of the shedding line on the hull has to be estimated from experiments.

Chapter 5 is divided into two parts. The first one deals with a vertical flat plate which can be considered representing an infinitely thin double model or an infinitely thin wing with a very small aspect ratio. In his thesis Casal had studied this problem, with the view to tackle a limiting case for the maneuvering ship. Casal's results were interesting. Here, we have examined some refinements for the boundary condition at the plate to be satisfied everywhere on its surface. But the problem has appeared more complicated than the one studied in Chapter 4. The second part of the Chapter is devoted again to the latter problem but the free surface effects are taken into account (the boundary condition there being linearized).

Chapter 6 is devoted to small motions about a uniform motion of translation. Two kinds of small motions are considered (1) those analogous to the small motions of a wing in the vertical plane and (2) those analogous to the small motions of a double model in the horizontal plane. The motions of the second kind are simpler with regard to the pressure continuity condition, but they are more complicated in that the vortex wake is necessarily three dimensional.

Chapter 7 combines several subjects which could not be attacked within the framework of a very general theory: interaction of the appendages with the hull, small motions of a wing profile, and problems more or less solved or understood (e.g., snaproll and pitchup of

submarines, effect of the history of the motion or equivalency of the apparent added masses on the course stability condition). Lastly, attention is drawn to the help that experimentation, simulation, and theory should bring to one another.

CHAPTER 1: NEEDS AND STATE OF THE ART

According to the French vocabulary, the "maneuverability" of a given ship is, in the mathematical sense of this term, the domain of all her possible steady motions. The term "handiness" characterizes the shortness of the time necessary to pass from a given steady motion to a second steady motion close to the first one.* Hence, "handiness" increases with the stability of the second motion. In the past, the opposite proposition was generally held.

Important results in the domain of directional stability have been obtained during the last few years or so by Contensou,[2] Davidson and Schiff,[3] Dieudonne,[4] and Grim.** I myself studied the stability of submarines in a vertical plane in a paper written 20 years ago, but not published.*** I had shown that if the stability becomes negative beyond a certain critical speed, the ship tends to experience periodic motions of considerable amplitude; these must be avoided at all costs. More recently, many other authors have contributed considerably to progress

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*For the distinction between "maneuverability" and "handiness," see Roy.[1]

**Unpublished reports of O. Grim, Hamburg Schiffbau - Versuchanstalt.

***Brard, R., "La tenue de plongée des sous-marins," prepared at the Bassin d'Essais des Carénes, Paris, in November 1953 but not published.

[1]Roy, M., "Le problème de la stabilité des regimes de vol," Bull. Ass. Tech. Mar. et Aéron., Paris (1931).

[2]Contensou, P., "Mécanique du Navire en route et en giration," Bull. Ass. Tech. Mar. et Aéron., Paris (1938).

[3]Davidson, K. S. M. and C. Schiff," Turning and Coursekeeping Qualities of Ships," Trans. Soc. Nav. Arch. Mar. Eng. (1946).

[4]Dieudonne, J., "Note sur la stabilité des régimes de route des navires," Bull. Ass. Tech. Mar. et Aéron., Paris (1949).

in the field of directional stability and control of bodies moving in water (see, for instance, the papers cited as references as well as the Proceedings of the International Towing Tank Conference (ITTC) and those presented at the seminar held in London during 1972).[5]

Optimization of the trajectories of a given ship should be the goal of research on maneuvering qualities. But this problem would be meaningless if the domain of all her possible study motions was too small. On the other hand, such an optimization is practically impossible if the equations of all the possible motions are not known accurately.

Unfortunately this is effectively the case. Because of the shortcomings of the theory, it is necessary to resort to experiments on captive and on free-running models. But to obtain good agreement between the measured trajectories and those derived from equations, it has become necessary to introduce a considerable number of terms into the equations. This leads to expansions whose uniqueness is questionable since equivalent predictions can be obtained by simultaneously altering several terms. Only the first terms of such expansions have physical meaning.

Theoretical research must be pursued actively in order to improve this rather poor situation. This is not an easy task for every theory implies assumptions.

In the present lectures, we consider only the case of liquid unbounded in all directions (except the vertical direction when a free surface exists) and we assume that the liquid is "almost" inviscid. This means that liquid adheres to the hull of the ship but that the other effects of viscosity are ignored. This considerably simplifies the problem and allows us to utilize vortex theory; thus, the system of hydrodynamic forces exerted on the body can include a lift. But the assumption prevents us from taking into consideration the phenomenon of separation due to strong adverse pressure gradients. Furthermore, the assumption does not provide us with means to determine the exact

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[5] Proceedings of the International Symposium on the Directional Stability and Control of Bodies Moving in Water, London (1972).

CHAPTER 2: MOTION OF A SOLID THROUGH AN UNBOUNDED LIQUID

This theory is classical;* it is reviewed here for comparison with results derived from the bound vortex theory.

We consider a right-handed system S of axes $O(x_1, x_2, x_3)$ moving with the body. Let u_1, u_2, u_3 denote the components on the moving axes of the velocity $\vec{V}_E(O)$ of O and u_4, u_5, u_6 those of the angular velocity $\vec{\Omega}$ of S. Let $\vec{I}$ be the momentum of the fluid, $\vec{J}$ the moment of the momentum about O, and $\vec{K}$ the moment of momentum about the fixed point O' with which O coincides at the time t under consideration. Let (ξ_1, ξ_2, ξ_3) and (ξ_4, ξ_5, ξ_6) be the components of $\vec{I}$ and $\vec{J}$, respectively.

As regards the force, the hydrodynamic forces exerted on the body are

$$(X_1, X_2, X_3) = -\frac{d\vec{I}}{dt}$$

and as regards the moment about O', they are

$$(X_4, X_5, X_6) = -\frac{d\vec{K}}{dt}$$

We may express the X's in terms of the ξ's. For instance,

$$X_1 = -\left(\frac{d\xi_1}{dt} + u_5\xi_3 - u_6\xi_2\right),$$

$$X_4 = -\left(\frac{d\xi_4}{dt} + u_5\xi_6 - u_6\xi_5 + u_2\xi_3 - u_3\xi_2\right)$$

On the other hand, we can show that the kinetic energy T is given by

$$T = \frac{1}{2}\rho \iint_\Sigma \Phi \frac{\partial\Phi}{\partial n} d\Sigma$$

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*After Kirchhoff, Kelvin, and Lamb analyses. For more detail, see Lamb.[6]

[6]Lamb, Sir Horace, "Hydrodynamics," Sixth Edition, Cambridge University Press (1932), Chapter VI.

position on the hull of the line of shedding of the free vortices. It follows that, if a slight change in its position can entail an important alteration of the calculated system of hydrodynamic forces exerted on the body, then it is necessary to determine the position of this line experimentally.

where Φ is the velocity potential inside the domain D_e exterior to the hull. It can also be written in the form:

$$T = \frac{1}{2} \sum\sum t_{ij} u_i u_j$$

where

$$t_{ij} = \rho \iint_{\Sigma} \phi_i \frac{\partial \phi_j}{\partial n} d\Sigma, \quad \phi_i = \Phi(u_1, u_2, \ldots u_6) \text{ for } u_i = 1 \text{ and}$$
$$u_j = 0 \text{ (every } j \neq i)$$

The t_{ij}'s are the elements of a square symmetric matrix T. They depend only on the geometry of the hull.

Since we can show that $\frac{\partial T}{\partial u_j} = \xi_j$, then by substituting the ξ_j's into the expresssions for the X's, we obtain the following Kirchhoff formulas:

$$\begin{cases} X_1 = -\frac{d}{dt}\left(\frac{\partial T}{\partial u_1}\right) - \left(u_5 \frac{\partial T}{\partial u_3} - u_6 \frac{\partial T}{\partial u_2}\right) \\ X_4 = -\frac{d}{dt}\left(\frac{\partial T}{\partial u_4}\right) - \left(u_5 \frac{\partial T}{\partial u_6} - u_6 \frac{\partial T}{\partial u_5} + u_2 \frac{\partial T}{\partial u_3} - u_3 \frac{\partial T}{\partial u_2}\right) \end{cases}$$

By solving six Neumann exterior problems, we can determine the ϕ_j and then the t_{ij}.

In the formula for X_i, the first term on the right side, namely $-\frac{d}{dt}\frac{\partial T}{\partial u_i}$, represents the effect of the inertia of the fluid, i.e., the effect of the so-called "added masses." The other terms give the expression for the hydrodynamic forces exerted on the body when its motion is uniform and therefore when the fluid motion is steady with

respect to the body. The resulting system of hydrodynamic forces acting on the body may thus be written in the form:

$$\mathcal{S}^{d} = \mathcal{S}_{in} + \mathcal{S}_{q.s.}$$

The part $\mathcal{S}_{in}$ in the contribution from the added masses and the part $\mathcal{S}_{q.s.}$ would exist alone if the motion of the body were uniform.

In a similar manner we can derive the system of forces from the kinetic energy T^{s} of the solid.

$$\mathcal{S}^{s} = \mathcal{S}^{s}_{in} + \mathcal{S}^{s}_{q.s.}$$

which should be exerted on it in order to define its motion by the six functions $u_1(t)$, $u_2(t)$, ... $u_6(t)$. In fact the system of forces exerted on the solid is the sum of $\mathcal{S}^{d}$, of the forces due to hydrostatic pressures on the hull and the system due to external forces. By expressing this sum as equivalent to the system $\mathcal{S}^{s}$, we obtain a set of six partial differential equations which are the equations of the motion of the solid.

THE D'ALEMBERT PARADOX

If $u_4 = u_5 = u_6 \equiv 0$, that is if $\vec{\Omega}_E \equiv 0$, and furthermore if u_1, u_2, u_3 are constants, we obtain

$$X_1 = X_2 = X_3 = 0$$

but, in general, the moment (X_4, X_5, X_6) is not zero. The system of hydrodynamic forces exerted on the solid reduces to a couple. This is the famous d'Alembert paradox.

STABLE AND UNSTABLE STEADY MOTIONS

Let us assume that the system of external forces is equivalent to zero. By specifying that $u_1, \ldots u_6$ are constants, we obtain the equations of all the steady motions of the solid. More particularly, we obtain the equations of the steady motions of translation. By doing this and studying the hydrodynamic couple exerted on the solid, Lamb showed that in the case of an ellipsoid whose axes coincide with its axes of inertia, the only stable translation is that in the direction of its smallest axis of symmetry.

Actually, however, experiments have shown that other translations can be stable. The explanation of this apparent contradiction is that the hydrodynamic forces on a body in a uniform motion of translation do not reduce to a couple alone. This is a consequence of the shedding of free vortices. We will see in Chapter 4 that this phenomenon alters both the quasi-steady system of forces and those due to inertial effects.

CHAPTER 3: MOTION OF A SOLID THROUGH AN UNBOUNDED, INVISCID FLUID ACCORDING TO THE BOUND VORTEX THEORY

KINEMATICS

Vortex Filaments, Vortex Tubes, Vortex Sheets

If the velocity $\vec{V}$ of a fluid is continuous and continuously differentiable in a certain domain D, we can define the vorticity $\vec{\omega}$ at every point M belonging to that domain. The vorticity at M is defined by

$$\vec{\omega}(M) = \text{curl } \vec{V}(M)$$

The properties of a given distribution $\mathcal{D} = (D,\vec{\omega})$ of $\vec{\omega}$ within D are derived from the Stokes theorem. If C is a closed circuit located within D, the circulation of $\vec{V}$ in that circuit is equal to the flux of $\vec{\omega}$ through any open surface S whose edge coincides with C:

$$\Gamma(C) = \int_C \vec{V}\cdot d\vec{s} = \iint_S \vec{\omega}\cdot\vec{n}\, dS$$

The unit vector $\vec{n}$ normal to S is in the positive direction with the sense selected on C.

A vortex filament L is tangent to $\vec{\omega}$ at every point, and a vortex tube T is the surface generated when the vortex filaments L intersect a given closed contour C. It follows from the above theorem that if C_1 reduces to C by a continuous deformation, then the circulation $\Gamma(C_1)$ in any closed contour C_1 on T is a constant equal to $\Gamma(C)$. $\Gamma(C)$ is termed the intensity of the tube T. A tube T can contain a unique filament L and its intensity can nevertheless be different from zero. In that case, $\vec{\omega}$ is infinite on L and $\vec{V}$ is no longer continuously differentiable at every point of D; L then has to be considered as an infinitely thin vortex tube.

A vortex sheet can also be defined by a limiting process. Such a sheet is a domain of very small thickness ε. The vorticity inside the sheet is $\vec{\omega} = \frac{\vec{T}}{\varepsilon}$; $\vec{T}$ is finite when ε goes to zero. The sheet reduces to a surface Σ and $\vec{T}$ is tangent to Σ at every point (see Figure 1).
Let P be a point belonging to Σ. Let Σ^+ and Σ^- be the two sides of Σ. Let $\vec{n}$ denote the unit vector normal to Σ in the direction from Σ^- toward Σ^+, and $(\vec{n}, \vec{\theta}, \vec{\tau})$ a right-handed system of three unit vectors, $\vec{\tau}$ being in the direction of $\vec{T}$. We may consider on Σ the vortex filaments L and the lines C tangent to $\vec{\tau}$ and $\vec{\theta}$, respectively, and define the elements of arc ds on L and $d\sigma$ on C so that $ds > 0$ in the direction of $\vec{\tau}$ and $d\sigma > 0$ in the direction of $\vec{\theta}$. The part of the sheet located between two lines $L(\sigma)$, $L(\sigma + d\sigma)$ is an infinitely flat tube of thickness ε and of intensity

$$d\Gamma = \vec{T}\cdot\vec{\tau}\ d\sigma$$

According to the Stokes theorem

$$d\sigma[\vec{\theta}(\vec{V}^+ - \vec{V}^-)]_P = d\Gamma$$

and

$$(\vec{V}^+ - \vec{V}^-)_P = (\vec{T} \wedge \vec{n})_P$$

Conversely, any surface through which $\vec{V}$ is discontinuous can be considered as the support of a vortex sheet.

Since the intensity of a vortex filament is a constant, a vortex filament cannot begin or end in the fluid. The support of L is a closed contour or its two ends are located on the boundary of the fluid domain (possibly at infinity).

The Poincaré and Biot and Savart Formulas

Let D_i denote the domain interior to a closed surface S and $\vec{n}$ the unit vector normal to S in the inward direction. Inside D_i, the velocity $\vec{V}$ is supposed to be continuously differentiable. By applying the classical formula

$$\text{curl } (\text{curl } \vec{A}) = \nabla(\text{div } \vec{A}) - \Delta\vec{A}$$

(where Δ is the Laplace operator) to the vector

$$\vec{A}(M) = \frac{1}{4\pi} \iiint_{D_i} \frac{\vec{V}(M')}{MM'} \, dD_i(M')$$

we can readily obtain the famous Poincaré formula:

$$\text{curl } \left[\frac{1}{4\pi} \iint_S \frac{(\vec{n} \wedge \vec{V})M'_i}{MM'} \, dS(M') + \frac{1}{4\pi} \iiint_{D_i} \frac{\vec{\omega}(M')}{MM'} \, dD_i(M') \right]$$

$$+ \nabla \left[- \frac{1}{4\pi} \iint_S \frac{(\vec{n} \cdot \vec{V})M'_i}{MM'} \, dS(M') - \frac{1}{4\pi} \iiint_{D_i} \frac{\text{div } \vec{V}(M')}{MM'} \, dD_i(M') \right]$$

$$= \begin{cases} \vec{V}(M) & \text{if } M \in D_i \\ 0 & \text{if } M \in D_e \end{cases} \qquad (3.1)$$

In this formula, D_e is the domain exterior to S, and M'_i is defined by $\overrightarrow{M'M'_i} = \vec{n}_{M'}(0^+)$.

In the following, we deal with liquids and div $\vec{V} \equiv 0$. Formula (3.1) gives the solution of the partial differential equation

$$\vec{\omega} = \text{curl } \vec{V}, \quad \text{inside } D_i$$

when $\vec{V}$ is given on the boundary S of D_i.

By means of a limiting process, it is seen that if S is at infinity, and if D_i contains a unique vortex filament L of intensity Γ, then

$$\vec{V}(M) = \text{curl } \frac{\Gamma}{4\pi} \int_L \frac{\overrightarrow{ds}(M')}{MM'} \tag{3.2}$$

This is the Biot-Savart formula.

Vortex Distribution Kinematically Equivalent to the Hull of a Body

Let Σ be the surface of the body and D_i, D_e the domains interior and exterior to Σ. Let Σ_i, Σ_e denote the two sides of Σ. The unit vector $\vec{n}$ normal to Σ is in the inward direction.

Our purpose is to show that there exists one vortex distribution inside D_i and on Σ which is equivalent to the body inside D_e and which satisfies the condition that the relative velocity is zero at Σ_i. To that end, we apply the Poincaré formula inside D_e. At infinity, $|\vec{V}| = 0\left(\frac{1}{R^3}\right)$; R is the distance from Σ. Since $\vec{n}$ is in the outward direction with respect to D_e and $\vec{\omega} \equiv 0$ inside D_e, we readily obtain:

$$\text{curl } \frac{1}{4\pi} \iint_\Sigma \frac{(-\vec{n}\Lambda\vec{V})M'_e}{MM'} d\Sigma(M') + \nabla \frac{1}{4\pi} \iint_\Sigma \frac{(\vec{n}\cdot\vec{V})M'_e}{MM'} d\Sigma(M')$$

$$= \begin{cases} \vec{V}(M) & \text{if } M \in D_e \\ 0 & \text{if } M \in D_i \end{cases}$$

Let $\vec{V}_E$ denote the velocity at a point moving with the body. The Poincaré formula applied to $\vec{V}_E$ inside D_i gives:

$$\operatorname{curl} \frac{1}{4\pi} \iint_{\Sigma} \frac{(\vec{n}\wedge\vec{V}_E)M'_i}{MM'} d\Sigma(M') + \nabla \frac{1}{4\pi} \iint_{\Sigma} \frac{(\vec{n}\cdot\vec{V}_E)M'_i}{MM'} d\Sigma(M')$$

$$+ \operatorname{curl} \frac{1}{4\pi} \iiint_{D_i} \frac{2\vec{\Omega}_E(M')}{MM'} dD_i(M') = \begin{cases} \vec{V}_E \text{ if } M \in D_i \\ 0 \text{ if } M \in D_e \end{cases}$$

By adding the two formulas, we have:

$$\operatorname{curl} \left[\frac{1}{4\pi} \iint_{\Sigma} \frac{(-\vec{n}\wedge\vec{V}_R)M'_e}{MM'} d\Sigma(M') + \iiint_{D_i} \frac{2\vec{\Omega}_E(M')}{MM'} dD_i(M') \right]$$

$$= \begin{cases} \vec{V}(M) \text{ if } M \in D_e \\ \vec{V}_E(M) \text{ if } M \in D_i \end{cases} \tag{3.3}$$

where $\vec{V}_R$ is the relative velocity $\vec{V} - \vec{V}_E$ of a fluid point. This shows that there exists a vortex distribution

$$\mathcal{D} = \left(\Sigma, \frac{\vec{T}}{\varepsilon}\right) + (D_i, 2\vec{\Omega}_E) \quad \text{with } \vec{T}(M') = (-\vec{n}\wedge\vec{V}_R)M'_e \tag{3.4}$$

which fulfills the requirement. Since function Γ is defined on Σ, we necessarily have

$$2\vec{\Omega}_E \cdot \vec{n} = - \frac{\partial^2\Gamma}{\partial\sigma\partial s} \tag{3.5}$$

Effective Determination of the Vortex Distribution

The vector $\vec{T}$ is determined by the condition that the left side of Equation 3.3 be equal to $\vec{V}_E$ when $M \varepsilon D_i$. This is satisfied if it is equal to $\vec{V}_E(M_i)$ when $M \varepsilon \Sigma_i$. This leads to the vectorial Fredholm equation of the second kind

$$-\frac{1}{2}(\vec{n}\wedge\vec{T})_M + \text{curl}\,\frac{1}{4\pi}\iint_{\Sigma}\frac{\vec{T}(P,\ t)}{MP}\,d\Sigma(P)$$

$$= \vec{V}_E(M,\ t) - \text{curl}\,\frac{1}{4\pi}\iiint_{D_i}\frac{2\vec{\Omega}_E(P,\ t)}{MP}\,dD_i(P) \tag{3.6}$$

This equation is singular for the vector $\vec{T}(M,\ t) = \lambda\cdot\vec{n}_M$, where λ is a constant, cancels the left side.

The discussion of the equation is rather tedious and it is simpler to proceed in another manner. First we replace $\mathcal{D}$ by the distribution

$$\begin{cases} \mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 \quad \text{with} \\ \mathcal{D}_1 = \left(\Sigma,\ \dfrac{\vec{T}_1}{\varepsilon}\right) + (D_i,\ 2\vec{\Omega}_E),\ \mathcal{D}_2 = \left(\Sigma,\ \dfrac{\vec{T}_2}{\varepsilon}\right) \end{cases} \tag{3.7}$$

The condition to be satisfied by $\vec{T}_1$ is that the velocity $\vec{V}_1$ induced by $\mathcal{D}_1$ is zero inside D_e. If this is possible, there exists within D_i a velocity potential Φ' such that

$$\vec{V}' = \vec{V}_E - \vec{V}_1 = \nabla\Phi' \quad (\text{within } D_i) \tag{3.8}$$

The discontinuity of $\vec{V}_1$ through Σ is tangent to Σ. Consequently,

$$\frac{\partial\Phi'}{\partial n} \equiv \vec{n}\cdot\vec{V}_E \quad \text{on } \Sigma_i \tag{3.9}$$

It follows that Φ' is the solution of a Neumann <u>interior</u> problem.

This solution exists since $\iint_\Sigma \vec{n}\cdot\vec{V}_E \, d\Sigma = 0$, and it is unique.

Conversely, when Φ' is known, $\vec{V}_1$ is determined inside D_i by Equation (3.8) and we have

$$\vec{T}_1(M) \equiv \vec{n}\Lambda\vec{V}_1(M_i) \quad \text{on } \Sigma \tag{3.10}$$

Let $\vec{V}_2$ be the velocity induced by $\mathcal{D}_2$. We have

$$\vec{V}_2 = \vec{V} \tag{3.11}$$

inside D_e (since $\vec{V}_1 \equiv 0$ in that domain). This velocity is irrotational also inside D_i. Hence $\mathcal{D}_2$ is equivalent to a normal doublet distribution on Σ:

$$\mathcal{D}_2 \sim (\Sigma, \mu'\vec{n}) \tag{3.12}$$

The density μ' of this distribution must be determined so that it generates inside D_i a velocity potential Φ_2 equal to Φ'; this entails

$$\vec{V} = \vec{V}_1 + \vec{V}_2 = \vec{V}_1 + (\vec{V}_E - \vec{V}_1) = \vec{V}_E \quad \text{within } D_i$$

It follows that

$$\Phi_2(M) = -\frac{-1}{4\pi}\iint_{\Sigma}\mu'(P)\frac{\partial}{\partial n_P}\frac{1}{MP}\,d\Sigma(P)$$

with

$$-\frac{1}{2}\mu'(M) - \frac{1}{4\pi}\iint_{\Sigma}\mu'(P)\frac{\partial}{\partial n_P}\frac{1}{MP}\,d\Sigma(P) = \Phi'(M_i), \quad \begin{cases} M \in \Sigma \\ \overrightarrow{MM_i} = \vec{n}_M(0+) \end{cases} \tag{3.13}$$

Now we have

$$\vec{T}_2(M)' = -\vec{n}_M \Lambda [\vec{V}(M_e) - \vec{V}'(M_i)] \tag{3.14}$$

The boundary condition on Σ_e, namely,

$$\vec{n}\cdot\vec{V} \equiv \vec{n}\cdot\vec{V}_E, \text{ on } \Sigma_e \tag{3.15}$$

is satisfied since the normal component of $\vec{V}_2$ is continuous through Σ and equal to $\vec{n}\cdot\vec{V}_E$ on Σ_i.

The above potential Φ_2 is the solution of a Dirichlet <u>interior</u> problem, and (3.13) has a unique solution. We have

$$\Phi(M_e) = \Phi_2(M_e) = \Phi_2(M_i) + \mu'(M) = \Phi'(M_i) + \mu'(M)$$

The solution of the problem is

$$\mathcal{D} = (\Sigma, \frac{\vec{T}}{\varepsilon}) + (D_i, 2\vec{\Omega}_E) \quad \text{with } \vec{T} = \vec{T}_1 + \vec{T}_2 \tag{3.16}$$

It follows from the definition of Φ' that at a point M located inside D_i and moving with the body, Φ' is a linear function of the

components $u_i(t)$. We have

$$\Phi'(M,t) = \sum_{j=1}^{6} u_j(t)\ \Phi_j(M) \tag{3.17}$$

where the functions Φ_j are independent of time. Furthermore, it also follows from (3.13) that μ' may be written in the form

$$\mu'(M,t) = \sum_{j=1}^{6} u_j(t)\ \mu_j'(M) \tag{3.18}$$

where the functions μ_j' are independent of time.

Generalization

It may be that the body is moving in an incident flow whose velocity is

$$\vec{V}_0(M,\ t) = \nabla\Phi_0(M,\ t) \tag{3.19}$$

In this case, the distribution D_2 becomes:

$$\mathcal{D}_2 \sim (\Sigma,\ (\mu' + \mu'')\vec{n}) \tag{3.20}$$

and the potential

$$\Phi_2 = \frac{-1}{4\pi} \iint_{\Sigma} (\mu' + \mu'')_{P,t} \frac{\partial}{\partial n_P} \frac{1}{MP}\, d\Sigma(P) \tag{3.21}$$

must satisfy the boundary condition

$$\Phi_2(M_i) = \Phi'(M_i) - \Phi_0(M_i) \quad \text{on } \Sigma_i. \tag{3.22}$$

Let G be the resolvent kernel of Equation (3.13). We have

$$\begin{cases} \mu'(M, t) = \sum_j u_i(t)\, \mu_j'(M) = \iint_\Sigma G(M, P)\, \Phi'(P, t)\, d\Sigma(P) \\ \\ \mu''(M, t) = \iint_\Sigma G(M, P)\, [- \Phi_0(P, t)[\, d\Sigma(P) \end{cases} \tag{3.23}$$

DYNAMICS

Forces Exerted on a Vortex Distribution

As in Chapter 2, we consider two right-handed systems of axes S, S', namely the system $0(x_1, x_2, x_3)$ which moves with the body and the system $0'(x_1', x_2', x_3')$ which is fixed in space. System S' is selected so that it coincides with S at a <u>certain</u> time t. Let M' be the position at t of the point M which moves with S, and $\vec{V}_E$, $\vec{\gamma}_E$ be respectively the velocity and the acceleration of M. Furthermore $\vec{V}(M, t)$ is the velocity of the fluid point P located at M at time t, and $\vec{V}_R = \vec{V} - \vec{V}_E$ is its relative velocity.

The force $\vec{F}$ per unit mass of the fluid is defined in system S' as

$$\vec{F} = \vec{F}(M', t)$$

According to the famous Helmholtz theorem, the vorticity $\vec{\omega}$ at a point M at time t is, in fact, a property of the fluid point P located at M at t. This follows from the Helmholtz equation.

$$\frac{\partial}{\partial t}\ \frac{\vec{\omega}}{\rho}\ (P,\ t) = \frac{d}{dt}\ \frac{\vec{\omega}}{\rho}\ (M,\ t) = \left(\frac{\vec{\omega}}{\rho}\cdot\nabla\ \ \vec{V}\right) \tag{3.24}$$

where ρ is the mass density of the fluid. For this equation to hold, it is necessary that the fluid be inviscid and that $\vec{F}$ be the gradient of a certain function $\mathcal{U}$ of M' and t:

$$\vec{F}(M',\ t) = \nabla\mathcal{U}(M',\ t) \tag{3.25}$$

If so, a vortex filament moves with the fluid and its intensity is independent of time. Consequently, no external force is exerted on the fluid points belonging to the vortex filament. One can say that the vortex filament is free.

Let us come back to the vortex distribution

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$$

which has been determined earlier so that it generates inside D_e a fluid motion identical with that due to the motion of the solid. The fluid points which belong to the distribution $\mathcal{D}$ do not move with the fluid. They do not satisfy Equation (3.24) because the adherence forces which keep them at rest with respect to the hull are due to some viscosity effect and do not meet the requirement of Equation (3.25).

The hydrodynamic forces exerted on the fluid points of $\mathcal{D}$ will be termed the hydrodynamic forces exerted on the distribution itself. This misuse of language is adopted for the sake of brevity. These hydrodynamic forces are obviously connected with the hydrodynamic forces exerted on the moving body itself. To determine the latter, we have to begin by determining the former.

Euler Equation in the Moving System of Axes

Let p be the pressure, p_s the hydrostatic pressure, and p_d the hydrodynamic pressure:

$$p_s = \rho U \ , \quad p_d = p - p_s = p - \rho U \tag{3.26}$$

Let $\vec{\gamma}_R$ be the relative acceleration and $\vec{\omega}_R$ the relative vorticity. Then,

$$\vec{\gamma}_R = \frac{d\vec{V}_R}{dt} \ , \quad \vec{\omega}_R = \text{curl}\ \vec{V}_R = (\vec{\omega} - 2\vec{\Omega}_E) \tag{3.27}$$

and by the Coriolis theorem, we have

$$\vec{\gamma} = \frac{d\vec{V}}{dt} = \vec{\gamma}_R + \vec{\gamma}_E + 2\vec{\Omega}_E \wedge \vec{V}_R$$

where $\vec{\gamma}_E = \dfrac{d\vec{V}_E}{dt}$. The Euler equation gives:

$$\begin{aligned}
\frac{1}{\rho}\nabla p_d &= -\left(\vec{\gamma}_R + \vec{\gamma}_E + 2\vec{\Omega}_E \wedge \vec{V}_R\right) \\
&= -\left[\frac{\partial \vec{V}_R}{\partial t} + \vec{\omega}_R \wedge \vec{V}_R + \nabla\left(\frac{1}{2} V_R^2\right)\right] - \left(\vec{\gamma}_E + 2\vec{\Omega}_E \wedge \vec{V}_R\right) \\
&= -\frac{\partial \vec{V}}{\partial t} + \left(\frac{\partial \vec{V}_E}{\partial t} - \vec{\gamma}_E\right) - \vec{\omega} \wedge \vec{V}_R - \nabla\left(\frac{1}{2} V_R^2\right)
\end{aligned}$$

Let Δ denote the axis of the helicoidal motion tangent to the motion of the system of axes S, and let $\vec{r}(M, t)$ be the vector orthogonal to Δ, with its origin on Δ and its end at M. The velocity of any point A located on Δ and moving with S is a vector $\vec{V}_E(A)$ parallel to Δ and

independent of the position of A on Δ. We obtain

$$\vec{V}_E(M) = \vec{V}_E(A) + \vec{\Omega}_E \wedge \vec{r}$$

$$\vec{\gamma}_E(M,\ t) = \frac{\partial \vec{V}_E}{\partial t}(M,\ t) + (\vec{V}_E(M)\cdot\nabla)\ \vec{V}_E(M)$$

and

$$(\vec{V}_E\cdot\nabla)\ \vec{V}_E = \vec{\Omega}_E \wedge [\vec{V}_E(A) + \vec{\Omega}_E \wedge \vec{r}]$$

$$= \vec{\Omega}_E \wedge (\vec{\Omega}_E \wedge \vec{r})$$

$$= -\ \Omega_E^2 \vec{r}$$

$$= \nabla(-\ \frac{1}{2}\ \Omega_E^2 r^2)$$

Hence the Euler equation becomes:

$$\frac{1}{\rho}\ \nabla p_d = -\ \frac{\partial \vec{V}}{\partial t}(M',\ t) - (\vec{\omega} \wedge \vec{V}_R)_{M,t} + \nabla(\ \frac{1}{2}\ \Omega_E^2 r^2)_{M,t} - \nabla(\ \frac{1}{2}\ V_R^2)_{M,t} \tag{3.28}$$

This equation reduces to

$$\frac{1}{\rho}\ p_d = -\ \frac{\partial \Phi}{\partial t}(M',\ t) + (\frac{1}{2}\ \Omega_E^2 r^2)_{M,t} - \frac{1}{2}\ V_R^2(M,\ t) \tag{3.29}$$

when

$$\vec{V}(M,\ t) = \nabla\Phi(M',\ t) \tag{3.30}$$

Hydrodynamic Forces on an Element of Vortex Sheet or on an Arc of Vortex Filament

Vortex Sheet. An element of vortex sheet is the domain dD bounded by two elements of surfaces $d\Sigma^+$ and $d\Sigma^-$; $d\Sigma^+$ is derived from $d\Sigma^-$ by a translation $\vec{n}\varepsilon$. Let de denote the set of fluid points located inside dE. Its momentum goes to zero with ε and so does its derivative. Hence, the system of forces exerted on dE is equivalent to zero. This system consists of the forces exerted by the adjacent sets of fluid points and of the <u>binding force</u> $d\vec{F}$. When ε goes to zero, we obtain

$$d\vec{F} - (p^+ - p^-)\,\vec{n}\,d\Sigma = 0 \qquad (3.31)$$

The binding force is thus normal to the surface Σ of the bound vortex sheet and the pressure is discontinuous through the sheet. If the vortex sheet is free, $d\vec{F} = 0$ and

$$p^+ = p^- \qquad (3.32)$$

on the sides of a free vortex sheet.

Vortex Filament. Let L_{b_1} be an arc of length δ_{s_1} included in the bound part L_b of a vortex filament L; see Figure 3. Let A denote the midpoint of L_{b_1}. The length δ_{s_1} is small enough for L_{b_1} to be regarded as a segment of the ζ - axis. Let (ξ, η, ζ) denote a moving, right-handed system of axes and (r, θ, ζ) a system of polar coordinates, r being the distance from the ζ - axis. In fact, L_{b_1} is the limit of a segment of vortex tube with a very small radius r_1; its intensity is

$$\vec{\Gamma} = \int_0^{2\pi}\int_0^{r_1} \vec{\omega}(r, \theta)\,r\,dr\,d\theta \qquad (3.33)$$

The relative velocity $\vec{V}_R$ may be written in the form

$$\vec{V}_R = \vec{V}_{R0} + (\vec{V}' - \overrightarrow{\delta V'}) + \overrightarrow{\delta V'} \tag{3.34}$$

where $\vec{V}_{R0}$ is due to all causes other than $\mathcal{L}$, $\vec{V}'$ to $\mathcal{L}$, and $\delta\vec{V}'$ to $\mathcal{L}_{b1}$. The momentum $\vec{I}(dE_1)$ of the set dE_1 of fluid points belonging to $\mathcal{L}_{b1}$ tends to zero with r_1 even when $\vec{\Gamma}$ remains finite. Let us put

$$\vec{V}_i = \lim_{r \downarrow 0} \vec{V}_{R0} + (\vec{V}' - \overrightarrow{\delta V'}) \tag{3.35}$$

This limit is the <u>incident</u> velocity on $\mathcal{L}_{b1}$. Let D' be the domain:

$$D' = \left\{ r,\ \theta,\ \zeta;\ 0 < r \leq R,\ 0 \leq \theta < 2\pi,\ -\frac{1}{2}\,\delta s_1 \leq \zeta \leq \frac{1}{2}\,\delta s_1 \right\} \tag{3.36}$$

This domain does not include the arc $\mathcal{L}_{b1}$. Let $\vec{I}(dE')$ be the momentum of the set dE' of fluid points located within D'. We have:

$$\frac{d}{dt}\,\vec{I}(dE') = \frac{d}{dt}\,\vec{I}(dE' + dE_1) = \rho \iint_S \vec{V}_R(\vec{n}\cdot\vec{V}_R)\, dS + O(\delta s_1 R^2)$$

where

$$S = S_L + \Sigma^+ + \Sigma^-\ ,\quad S_L = \left\{ r = R,\ \theta \varepsilon \left[0,\ 2\pi\right),\ |\zeta| < \frac{1}{2}\,\delta s_1 \right\}$$

$$\Sigma^{\pm} = \left\{ 0 \leq r \leq R\ ,\ \theta \varepsilon \left[0,\ 2\pi\right),\ \zeta = \pm\,\frac{1}{2}\,\delta s_1 \right\}$$

while $\vec{n}$ is normal to S in the outward direction.

Let u_i, v_i, w_i denote the components of $\vec{V}_i$ in the ξ, η, and ζ directions, respectively. We have

$$\lim_{R\downarrow 0} \frac{d}{dt} \vec{I}(dE') = \lim_{R\to 0} \rho \frac{\Gamma}{2\pi} \int_0^{2\pi} \left(-\frac{v_i}{R} \sin^2\theta \, \vec{i}_x + \frac{u_i}{R} \cos^2\theta \, \vec{i}_y \right) R \, d\theta \, \delta s_1 \tag{3.37}$$

$$= \frac{1}{2} \rho \, \vec{\Gamma} \wedge \vec{V}_i \, \delta s_1$$

Now, let $-d\vec{F}_T$ denote the force exerted by dE_1 on dE'. By the momentum theorem, we have

$$\frac{d}{dt} \vec{I}(dE') = -d\vec{F}_T + \iint_S - p\vec{n} \, dS \tag{3.38}$$

From (3.28), we derive

$$\frac{1}{\rho} \nabla p_d = -\frac{\partial \vec{V}_i}{\partial t} - \left(\text{curl } \vec{V}_{R_0} \right) \wedge (\vec{V}_i + \delta\vec{V}') - \nabla \left[\frac{1}{2} (\vec{V}_i + \delta\vec{V}')^2 \right] \quad \text{within } D' \tag{3.39}$$

When R decreases, $\vec{V}_{R_0}$, $\vec{V}_i$, $\frac{\partial \vec{V}_i}{\partial t}$ can be considered constant vectors and the first three terms of Equation (3.39) contribute nothing to $\frac{d}{dt} \vec{I}(dE')$. Furthermore,

$$\delta\vec{V}' \simeq \frac{\vec{\Gamma}}{2\pi r^2} \wedge \vec{r} \tag{3.40}$$

Hence

$$\frac{1}{\rho} p_d \simeq - \frac{\Gamma}{2\pi r^2} \vec{V}_i \cdot (\vec{i}_\zeta \Lambda \vec{R}) - \frac{1}{2} \left(\frac{\Gamma}{2\pi R^2} \right)^2 \quad \text{on } S_L$$

The contributions from Σ^+ and Σ^- cancel each other by symmetry. We thus obtain

$$\lim_{R \downarrow 0} \frac{d}{dt} \vec{I}(dE') = - d\vec{F}_T - \frac{1}{2} \rho \, \vec{\Gamma} \Lambda \vec{V}_i \, \delta s_1 \tag{3.41}$$

Finally, comparing with Equation (3.37), we have

$$\boxed{d\vec{F}_T = - \rho \, \vec{\Gamma} \Lambda \vec{V}_i \, \delta s_1} \tag{3.42}$$

This formula yields the expression for the force exerted by the incident flow on an arc δs_1 of a bound vortex filament. It can be considered *as the Kutta-Joukowski theorem.*

System of Forces Associated with the Vortex
Distribution $\mathcal{D}$ Kinematically Equivalent to
the Moving Body.

The vortex distribution kinematically equivalent to the moving body is taken in the form (3.7). Its properties are described by equations (3.8)-(3.23). The fluid points belonging to $\mathcal{D}$ do not move with the liquid, but with the body. Consequently, Helmholtz's theorem does not apply. The vortex filaments whose union makes the vortex distribution and the fluid taking part in the general motion exert a mutual action on each other.

We shall first assume that $u_1, u_2, \ldots, u_6$ are constants and therefore that the relative motion is steady.

Let Σ_e, Σ_i be the external and internal sides of the hull surface Σ, respectively. The fluid points belonging to the vortex sheet are located between Σ_i and Σ_e. Let M be a point on Σ, and M_i, M_e the points located on Σ_i and on Σ_e and defined by

$$\overrightarrow{MM}_i = \vec{n}_M(0+), \quad \overrightarrow{MM}_e = -\vec{n}_M(0+). \tag{3.43}$$

The incident velocity on the set dE of fluid points located inside the volume $d\Sigma \times \varepsilon$ is

$$\vec{V}_i(M) = \frac{1}{2}\,\vec{V}_R(M_e), \tag{3.44}$$

where $\vec{V}_R$ is the relative velocity. According to (3.31) and (3.42), the hydrodynamic force exerted on dE is given by

$$\begin{aligned}\overrightarrow{dF}_T = (p_d(M_e) - p_d(M_i))\vec{n}_M\, d_\Sigma(M) &= -\,\rho\,\vec{T}(M) \wedge \vec{V}_i(M)d\Sigma(M)\\ &= -\,\frac{1}{2}\,\rho V_R^2\,(M_e)\,\vec{n}_M\,d\Sigma(M)\end{aligned} \tag{3.45}$$

Let S_T denote the system of forces dF_T. We write symbolically

$$S_T = (\Sigma,\ d\vec{F}_T). \tag{3.46}$$

Similarly, the system of hydrodynamic forces exerted on the body may be written as

$$S_d = (\Sigma_e,\ p_d(M_e)\,\vec{n}\,d\Sigma_e) \tag{3.47}$$

Inside D_i, the u_i's being constant, we have:

$$\frac{1}{\rho}\,\nabla p_d = -\vec{\gamma}_E = \Omega_E^2\,\vec{r} = \nabla\left(\frac{1}{2}\,\Omega_E^2\,r^2\right) \tag{3.48}$$

whence

$$p_d\,(M) = \frac{1}{2}\,\rho\,\Omega_E^2\,r^2 + \text{constant} \tag{3.48'}$$

The system of forces exerted on the vortex sheet by the liquid located inside D_i is

$$(-p_d(M_i)\vec{n}_M\,d\Sigma_i) = (-\rho\Omega_E^2\,\vec{r}(C)dD_i) = -\,m\Omega_E^2\,\vec{r}_C = -\,\vec{F}_C \tag{3.49}$$

C being the center of volume of D_i, $m = \rho D_i$ the mass of the body, and $\vec{r}_C$ the vector with its origin is on the axis of helicoidal motion of the body and its end at C. The system (3.49) reduces to a unique force, namely the centripetal force acting on the mass m.

One observes that

$$p_d(M_e) = \frac{1}{2}\,\rho\Omega_E^2\, r^2(M) - \frac{1}{2}\, V_R^2\,(M_e), \tag{3.50}$$

which is in agreement with (3.29) since the relative motion is steady ($\frac{\partial\Phi}{dt}$ = 0 at any point fixed with respect to the body and located inside D_e).

We have seen that the velocity potential Φ inside D_e is identical with the potential Φ_2 due to a normal doublet distribution of density μ' on Σ. This density is determined by the Fredholm equation expressing that the interior determination of Φ_2 coincides with the velocity potential Φ' such that $\nabla\Phi' = \vec{V}_E - \vec{V}_1$ (equation (3.8)). We have, in general,

$$\Phi'(M, t) = \sum_j u_j(t)\,\Phi'_j(M) \tag{3.17}$$

and

$$\Phi(M_e, t) = \Phi'(M_i, t) + \mu'(M, t)$$

with

$$\mu'(M, t) = \sum_j u_j(t)\,\mu'_j(M) \tag{3.18}$$

The vortex theory gives means for calculating S_d since the six potentials Φ_j's follow from the solution of the Neumann interior problems expressing that

$$\frac{\partial\,\Phi'_j}{\partial n}(M_i, t) = \vec{n}\cdot\vec{V}_{Ej} \text{ on } \Sigma_i \tag{3.51}$$

and from the six Dirichlet interior problems expressing that

$$\Phi_j(M_i, t) = \Phi'_j(M_i, t) \text{ on } \Sigma_i \tag{3.52}$$

Let us consider now the case when the motion of the body is not uniform. Equations (3.48) - (3.50) no longer hold. But (3.50) has to be replaced by (3.29) and one has

$$\frac{\partial\Phi}{\partial t}(M_e, t) = \sum_j \dot{u}_j(t)\,(\Phi'_j(M_i) + \mu'_j(M)) \tag{3.53}$$

where $\dot{u}_j(t) = \frac{d}{dt} u_j(t)$. This gives

$$S_d = S_{in} + S_T - \vec{F}_C \tag{3.54}$$

with

$$S_{in} = (-\rho \sum_j \dot{u}_j \{\Phi'_j(M_i) + \mu'_j(M)\} \vec{n}_M \, d\Sigma(M)) \tag{3.55}$$

$$S_T = (-\rho \frac{1}{2} V_R^2(M) \vec{n}_M \, d\Sigma(M)) \tag{3.56}$$

$$\vec{F}_C = -\frac{1}{2} \Omega_E^2 \vec{r}_C^{\,2} \tag{3.57}$$

S_{in} is the system of forces due to the so-called "added masses". The system

$$S_T - \vec{F}_C = -\frac{1}{2} [\rho \ V_R^2 \ (M) \vec{n}_M \, d\Sigma \ (M) + \Omega_E^2 \vec{r}_C^{\,2}] \tag{3.58}$$

is the quasi-steady system of forces that is the one obtained by neglecting the acceleration of the body.

The reason why equations (3.48) - (3.49) may no longer hold is that $\vec{\gamma}_E$ may be rotational. To maintain at rest the liquid located inside D_i, it is necessary to add inside D_i a force $\vec{F}'$ per unit mass, so that

$$\nabla \wedge (\vec{F}' - \vec{\gamma}_E) \equiv 0. \tag{3.59}$$

Since

$$\vec{\gamma}_E = \frac{\partial \vec{V}_E}{\partial t} + \nabla(\frac{1}{2} \Omega_E^2 r^2),$$

condition (3.59) implies the existence of a potential ψ such that

$$\vec{F}' - \vec{\gamma}_E = \nabla \frac{\partial \psi}{\partial t} \tag{3.60}$$

This gives

$$p_d(M_i, t) = \rho \frac{\partial \psi}{\partial t} (M_i, t) + \frac{1}{2} \rho \Omega_E^2 r^2(M, t) \text{ on } \Sigma_i$$

whence

$$\frac{\partial \psi}{\partial t} (M_i, t) = - (\frac{\partial \Phi}{\partial t})_{M_e}, t. \tag{3.61}$$

One may select for ψ a harmonic function and write

$$\psi (M, t) = \sum_j u_j(t) \psi_j(M) \text{ inside } D_i$$

The ψ_j's are solutions of the Dirichlet interior problems

$$\psi_j(M_i) = - \Phi_j(M_e), \quad (j = 1, \ldots 6) \tag{3.62}$$

Equations (3.59) - (3.63) are not of great interest and are given for the sake of completeness only. The equations of importance are (3.54) - (3.57).

Generalization

Let us suppose that the liquid motion is due not only to the motion of the body alone but also to other causes. The velocity generated by these other causes is irrotational inside D_i, and the kinematic solution is obtained as indicated in the section on effective calculation of the vortex distribution. The two systems $S_{q.s}$ and S_{in} are altered. The new system of bound vortices is

$$\text{with} \quad \begin{cases} \mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}'_2 \\ \mathcal{D}_2 = \left(\Sigma, \dfrac{\vec{T}'_2}{\varepsilon}\right) = (\Sigma, \mu' \vec{n}), \\ \mathcal{D}'_2 = \left(\Sigma, \dfrac{\vec{T}_2}{\varepsilon}\right) \simeq (\Sigma, \mu''\vec{n}) \end{cases}$$

The velocity in the part of D_e close to Σ_e is, in general, irrotational.* The velocity potential in this region is

$$\Phi(M, t) = \Phi'_2(M, t) + \Phi''_2(M, t) + \Phi_0(M, t)$$

with

$$\Phi'_2(M_i, t) + \Phi''_2(M_i, t) = - \frac{1}{4\pi} \iint_\Sigma (\mu' + \mu'')_{P,\, t} \frac{\partial}{\partial n_P} \frac{1}{MP} \, d\Sigma(P)$$

$$= \Phi'(M_i, t) - \Phi_o(M_i, t)$$

From the Kutte-Joukowski theorem we have

$$d\vec{F}_T = - \frac{1}{2} \rho \, (\vec{T}_1 + \vec{T}'_2 + \vec{T}''_2) \wedge (\vec{V}'_R + \vec{V}''_R) \, d\Sigma(M) \tag{3.63}$$

with

$$\vec{V}'_R = \nabla\Phi'_2 + \nabla\Phi''_2 - \vec{V}_E$$

The system of forces $S_{q.s}$ becomes

$$S_{q.s} = [d\vec{F}_T] - \vec{F}_c(t) \tag{3.64}$$

where $d\vec{F}_T$ is given by Equation (3.63). The system due to the true and apparent added masses becomes

$$S_{in} = [S'_{in} + S''_{in}] \tag{3.65}$$

where S'_{in} is the system S_{in} of the preceding section and S''_{in} is

$$S''_{in} = [-\rho \frac{\partial \mu''}{\partial t} (M, t)\vec{n}_M \, d\Sigma(M)] \tag{3.66}$$

S''_{in} is the system due to the "apparent added masses."

The above equations for non-steady motions imply the assumption that no streamwise vortices are shed by the body. In the opposite case, if the motion is unsteady, the incident velocity induced by these vortices varies with time and new bound vortices appear on the hull surface. This phenomenon will be studied in Chapter 6.

CHAPTER 4: EXTENSION OF THE LIFTING SURFACE THEORY TO BODIES WITH FINITE DISPLACEMENTS

For the sake of brevity, the present chapter considers mainly translational motion of bodies. The theory expounded in Chapter 3 provides the necessary tools when the angular velocity varies. The problem of small motions about a uniform translation will be examined in Chapter 6 and the effect of the angular velocity will not be ignored. We begin with thin wings.

THIN WINGS

As before, let S denote the right-handed system of axes moving with the body. The wing is generally symmetric with respect to the plane $y = 0$; it is generated by profiles $\mathcal{P}(y)$, and its transverse sections are generated by the planes y = constant, with $-Y \leq y \leq Y$. The length $\ell(y)$ of $\mathcal{P}(y)$ is the difference $X(B) - X(A)$ of the abscissas of the ends B and A of the profile. The base chord L of the wing is the length of $\mathcal{P}(0)$. The wing moves in the negative x-direction. When y varies from $-Y$ to $+Y$, the lines described by points A and B are termed leading edge and trailing edge, respectively. The edges are not necessarily straight lines parallel to the y-axis, and, for $y = \pm Y$, points A and B do not necessarily coincide. Let $z^+(x, y)$ and $z^-(x, y)$ be the third coordinates of a point respectively describing the upper arc AB and the lower arc AB of the profile $\mathcal{P}(y)$. The chord of the profile $\mathcal{P}(y)$ is the length ℓ of a straight line joining A to B. The chords of the profiles are not necessarily parallel to one another. The curvature of a profile $\mathcal{P}(y)$ is generally very large at B, and so this point can be considered an edge in the geometrical sense of the word. In contrast, the profile is rounded at A. In all the cases, the surface Σ of the wing is defined by the equations $z = z^+(x, y)$ and $z = z^-(x, y)$. The aspect ratio of the wing is the ratio Y/L; it is rarely less than 5 or 6 and can be much greater.

The angle of attack is the angle $(\overrightarrow{AB}, Ox)$ and it is generally small. In this case, the streamlines of the relative motion leave the wing from

the trailing edge and there is no separation except when the curvature of the leading edge is too large.

Although the motion is three-dimensional, it is close to two-dimensional in the vicinity of the plane y = 0 when the aspect ratio is sufficiently large. Two-dimensional motions can be realized in wind tunnels. In this case, the wing is the part of a cylindrical surface that is parallel to the y-axis and intersects the walls $y = \pm Y$ of the wind tunnel. Experiments show that the circulation Γ of the velocity in closed circuits surrounding the wing is not zero and that the relative streamlines leave the profiles at B. These two facts are interconnected. They constituted an enigma for many years because they seemed to contradict a consequence of the Helmholtz theorem according to which the vorticity in a fluid motion is zero when the fluid starts from rest without any shock. This question was elucidated by Prandtl who demonstrated experimentally that a free vortex with an intensity equal and opposite to $\frac{d\Gamma}{dt}$ dt appears at the trailing edge and is carried by the flow to infinity downstream from the wing.*

When the aspect ratio of the wing is finite, a free vortex sheet is shed from the trailing edge $\mathcal{B}$. Let Σ_f denote the surface of this free vortex sheet. When the relative motion is steady, the free vortex sheet is unbounded in the positive x-direction. It is bounded ahead by $\mathcal{B}$ and its edges are the streamlines coming from the two lateral edges of the wing. The continuity of the pressure through Σ_f entails $\vec{V}_R^+ - \vec{V}_R^- = \vec{n}_f \wedge \vec{T}_f$ where $\vec{n}_f$ is the unit vector normal to Σ_f in the direction from Σ_f^+ toward Σ_f^-; here Σ_f^+ and Σ_f^- are the upper and lower sides of Σ_f, respectively. The velocity $\vec{V}$ of the fluid points with respect to a fixed system of axes S' is small, and $\vec{T}_f$ is approximately parallel to $-\vec{V}_E$. In fact, the angle $(0x, \vec{V}_R^+)$ is slightly greater than the angle $-(0x, \vec{V}_R^-)$, and at a large distance from the wing, the free vortex filaments L_f tend to wind around the free tip vortices coming from the lateral edges of the

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*Reported to the Third Congress of Applied Mechanics, Stockholm (1931).

wing. The vortex filament L_f is the free part of vortex filaments $L = L_f + L_b$, the bound arcs L_b being located on Σ.

When the wing is very thin, the bound vortices $\frac{\vec{T}_b^+}{\varepsilon}$ on Σ^+ and $\frac{\vec{T}_b^-}{\varepsilon}$ on Σ^- at points $(x, y, z^+(x, y))$ and $(x, y, z^-(x, y))$ can be replaced by a unique vortex $\frac{\vec{T}_b^+}{\varepsilon}$ located on the skeleton Σ_0 of the wing. In this case, Σ_0 is considered as a vortex sheet. The tangential component of the velocity is discontinuous through Σ_0. We can write

$$\vec{V}(M) = \nabla\Phi(M), \quad \Phi(M) = \frac{-1}{4\pi} \iint_{\Sigma_0+\Sigma_f} \mu(P) \frac{\partial}{\partial n_p} \frac{1}{MP} \, d\Sigma(P)$$

It is generally assumed that Σ_f is the surface generated by half-straight lines whose origins are on the trailing edge $\mathcal{B}$ and which are parallel to the x-axis. The function μ_f with which μ coincides on Σ_f is constant on such a generatrix. Inside Σ_0, the lines μ_b = constant are orthogonal to the relative streamlines. Let $(\vec{n}, \vec{\theta}, \vec{\tau})$ be a right-handed system of three unit vectors; $\vec{n}$ is normal to Σ_0, in the direction from Σ_0^+ toward Σ_0^- and $\vec{\tau}$ is in the direction of $\vec{T}_b$. Then $\vec{V}_R$ is in the direction of $\vec{\theta}$. In the linearized theory,

$$\frac{1}{\rho}\,\delta p = \frac{1}{\rho}\,(p^- - p^+) = |V_E| \frac{\partial \mu_b}{\partial \sigma} \quad , \text{(with } d\vec{\sigma} = \vec{\theta}\, d\sigma)$$

and the lift coefficient is

$$C_L \simeq \frac{1}{\Sigma_0 V_E^2} \iint_{\Sigma_0} V_E \frac{\partial \mu_b}{\partial \sigma} \, d\Sigma_0 = \frac{1}{\Sigma_0 V_E} \int_C \mu \, ds$$

where Σ_0 is the area of Σ_0 and C the contour of the wing.

The boundary conditions are

$$\begin{cases} \vec{V} \cdot \vec{n} = \vec{V}_E \cdot \vec{n} \quad \text{on } \Sigma_0 \\ \mu = \text{constant on the leading edge} \\ \dfrac{\partial \mu}{\partial \sigma} = 0 \text{ on the trailing edge} \end{cases}$$

The third condition states that there is no pressure jump on Σ_0 through the trailing edge. This is the expression for the Kutta condition in three-dimensional motions. It is known that the above conditions determine μ_b on Σ_0. We will verify this point in Chapter 5.

The leading edge is difficult to determine because the relative velocity is very high in this region when the shock-free condition is not fulfilled (that is, when the relative velocity is not tangent to Σ_0 along the leading edge). The theory of matched asymptotic expansions is generally used to circumvent the difficulty.

We conclude this portion with a short discussion of the theory of thin wings; note that in this theory the bound vortex filaments on the skeleton are all open. All the other cases that will be examined later have a subclass of vortex filaments closed on the surface of the body.

WINGS WITH A FINITE THICKNESS

The free vortex filaments $\mathcal{L}_f$ are arcs of vortex filaments $\mathcal{L} = \mathcal{L}_b \cup \mathcal{L}_f$, $\mathcal{L}_b$ being the bound arc of $\mathcal{L}$. Consequently the vortex distribution $\mathcal{D}$, which is kinematically equivalent to the set consisting of the wing and the free vortex sheet, can be written in the form

$$\begin{cases} \mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 \quad \text{with} \\ \mathcal{D}_1 = \left(\Sigma, \dfrac{\vec{T}_1}{\varepsilon}\right) + (D_i, 2\vec{\Omega}_E) \\ \mathcal{D}_2 = \left(\Sigma + \Sigma_f, \dfrac{\vec{T}_2}{\varepsilon}\right) \end{cases} \tag{4.1}$$

The vortex distribution $\mathcal{D}_1$ generates the velocity

$$\begin{cases} \vec{V}_1 \equiv 0 \quad \text{inside } D_e \\ \text{curl } \vec{V}_1 \equiv 2\vec{\Omega}_E \quad \text{within } D_i \end{cases} \tag{4.2}$$

As in Chapter 3, we have

$$\vec{V}_1(M, t) = \sum_j u_j(t)\vec{V}_{1j}(M) \quad \text{within} \quad D_i, \ \vec{n}\cdot\vec{V}_1 \equiv 0 \quad \text{on} \quad \Sigma_i \tag{4.3}$$

and there exists a velocity potential Φ' defined within D_i such that

$$\begin{cases} \vec{V}_E - \vec{V}_1 = \nabla\Phi' \quad \text{with} \quad \dfrac{\partial\Phi'}{\partial n} = \vec{n}\cdot\vec{V}_E \quad \text{on} \quad \Sigma_i \\ \Phi'(M, t) = \sum_j u_j(t)\Phi'_j(M) \quad \text{within} \quad D_i \end{cases} \tag{4.4}$$

On the other hand, the vortex distribution $\mathcal{D}_2$ is equivalent to a normal doublet distribution on $(\Sigma + \Sigma_f)$:

$$\mathcal{D}_2 \simeq (\Sigma, \mu'\vec{n}) + (\Sigma_f, \mu_f\vec{n}_f) \tag{4.5}$$

Here $\vec{n}$ is normal to Σ and $\vec{n}_f$ to Σ_f. Let us put

$$\begin{cases} \Phi_2' = \dfrac{-1}{4\pi} \displaystyle\iint_\Sigma \mu'(P, t) \frac{\partial}{\partial n_p} \frac{1}{MP} \, d\Sigma(P) \\ \\ \Phi_f = \dfrac{-1}{4\pi} \displaystyle\iint_\Sigma \mu_f(P, t) \frac{\partial}{\partial n_p} \frac{1}{MP} \, d\Sigma(P) \end{cases} \tag{4.6}$$

The total velocity induced by $\mathcal{D}$ is

$$\vec{V} = \begin{cases} \vec{V}_2' + \vec{V}_f & \text{inside } D_e \\ \\ \vec{V}_1 + \vec{V}_2' + \vec{V}_f & \text{inside } D_i \end{cases} \tag{4.7}$$

Let us determine μ' by the condition

$$\Phi_2' = \Phi' - \Phi_f \tag{4.8}$$

within D_i. We obtain

$$\vec{V} = \vec{V}_1 + (\nabla\Phi' - \nabla\Phi_f) + \nabla\Phi_f = \vec{V}_1 + \vec{V}_E - \vec{V}_1 = \vec{V}_E \tag{4.9}$$

within D_i.

Since the normal derivatives of Φ_2' and Φ_f are continuous through Σ, and as $\vec{n}\cdot\vec{V} \equiv \vec{n}\cdot\vec{V}_E$ on Σ_i, the boundary condition

$$\vec{n}\cdot\vec{V} \equiv \vec{n}\cdot\vec{V}_E \quad \text{on} \quad \Sigma_e \tag{4.10}$$

is satisfied. To determine μ_2' on Σ, it is necessary to know $\Phi_f(M, t)$. This implies that $\Sigma_f(t)$ and $\mu_f(P, t)$ are known functions of t. Before examining this problem, it is suitable to study in some detail the characteristics of the vortex sheet in the neighborhood of the trailing edge.

CHARACTERISTICS OF THE FLOW IN THE NEIGHBORHOOD OF THE TRAILING EDGE (See Figures 4 and 5.)

The trailing edge $\mathcal{B}$ divides the region of the wing close to it into two parts, Σ_1 which is included in the pressure side and Σ_2 which is included in the suction side. Let Σ_{f_2} be the side of Σ_f which corresponds to Σ_2 and let Σ_{f_1} be the side corresponding to Σ_1. The unit vector $\vec{n}$ normal to Σ is in the inward direction. On Σ_f, the unit vector $\vec{n}_f$ normal to Σ_f is directed from Σ_{f_2} to Σ_{f_1}.

Consider three vortex filaments L_f, L_1, L_2 located on Σ_f, Σ_1 and Σ_2, respectively, which intersect the trailing edge $\mathcal{B}$ at the same point B; M_f, M_1, and M_2 are three points close to $\mathcal{B}$ on L_f, L_1, L_2, respectively (see Figure 4). Furthermore consider three pairs of points, namely,

$$(M_{f_1}, M_{f_2});\ (M_{i_1}, M_{e_1});\ (M_{i_2}, M_{e_2})$$

defined by

defined by

$$\overrightarrow{M_f M_{f_1}} = \vec{n}_f(0^+) \quad ; \quad \overrightarrow{M_f M_{f_2}} = -\vec{n}_f(0^+)$$

$$\overrightarrow{M_1 M_{i_1}} = \vec{n}_{M_1}(0^+) \quad ; \quad \overrightarrow{M_1 M_{e_1}} = -\vec{n}_{M_1}(0^+)$$

$$\overrightarrow{M_2 M_{i_2}} = \vec{n}_{M_2}(0^+) \quad ; \quad \overrightarrow{M_2 M_{e_2}} = -\vec{n}_{M_2}(0^+)$$

Let C be the closed circuit

$$C:\ M_{e_2}\ M_{f_2}\ M_{f_1}\ M_{e_1}\ M_{i_1}\ M_{i_2}\ M_{e_2}$$

Let η be the order of magnitude of the distance from B of the points M_f, M_1, and M_2. The circulation $\Gamma(C)$ of $\vec{V}$ in C is

$$\Gamma(C) = [\Phi(M_{e_2}) - \Phi(M_{i_2})] + [\Phi(M_{f_1}) - \Phi(M_{f_2})] + \Phi(M_{i_1}) - \Phi(M_{e_1}) + O(\eta) = 0$$

Hence, since μ_1' and μ_2' represent the determination of μ' on Σ_1 and Σ_2, respectively, we have:

$$\mu'_2 - \mu_f - \mu'_1 = 0 \text{ or } \mu_f = \mu'_2 - \mu'_1 \text{ at B} \tag{4.11}$$

Let L_f', L_1', L_2' be three other vortex filaments on Σ_f, Σ_1, and Σ_2, intersecting $\mathcal{B}$ at the same point B'. Let M_1', M_2', and M_f' be three points on the relative streamlines containing M_1, M_2, and M_f, respectively. We introduce the following four closed circuits:

$$C_1 : M_{e_1} \quad M'_{e_1} \quad M'_{f_1} \quad M_{f_1} \quad M_{e_1}$$

$$C_2 : M'_{e_2} \quad M_{e_2} \quad M_{f_2} \quad M'_{f_2} \quad M'_{e_2}$$

$$C_i : M_{i_1} \quad M'_{i_1} \quad M'_{i_2} \quad M_{i_2} \quad M_{i_1}$$

$$C_f : M_{f_1} \quad M'_{f_1} \quad M'_{f_2} \quad M_{f_2} \quad M_{f_1}$$

The three pairs (M'_{f_1}, M'_{f_2}) (M'_{i_1}, M'_{e_1}) and (M'_{i_2}, M'_{e_2}) are derived from M'_f, M'_1, M'_2 as the first three pairs are derived from M_f, M_1, M_2. The flux of $\vec{T}_2$ through C_1 is zero and also through C_2 and through C_i since no vortex filament intersects open surfaces bounded by these contours. Hence Γ_2 being the circulation of the velocity $\vec{V}_2 = \vec{V}'_2 + \vec{V}_f$ due to $\mathcal{D}_2 = \left(\Sigma + \Sigma_f, \frac{\vec{T}_2}{\varepsilon} \right)$, we have

$$\Gamma_2(C_1) = 0 \ , \quad \Gamma_2(C_2) = 0 \ , \quad \Gamma_2(C_i) = 0 \tag{4.12}$$

Through C_f, the flux of $\vec{T}_f$ is

$$\Gamma_2(C_f) = [\vec{V}_R(M_{f_1}) - \vec{V}_R(M_{f_2})] \cdot \overrightarrow{M_f M'_f} \tag{4.13}$$

From Equations (4.12) we derive:

$$\vec{V}_2(M_{e_1}) \cdot \overrightarrow{M_1 M'_1} + \vec{V}_2(M_{f_1}) \cdot \overrightarrow{M'_f M_f} = O(\eta)$$

$$\vec{V}_2(M_{e_2}) \cdot \overrightarrow{M'_2 M_2} + \vec{V}_2(M_{f_2}) \cdot \overrightarrow{M_f M'_f} = O(\eta)$$

$$\vec{V}_2(M_{i_1}) \cdot \overrightarrow{M_1 M'_1} + \vec{V}_2(M_{i_2}) \cdot \overrightarrow{M'_2 M_2} = O(\eta)$$

By adding the first two equations and subtracting the third one, we obtain:

$$[\vec{V}_2(M_{e_1}) - \vec{V}_2(M_{i_1})]\ \overrightarrow{M_1M_1'} + [\vec{V}_2(M_{e_2}) - \vec{V}_2(M_{i_2})]\ \overrightarrow{M_2'M_2} = \Gamma_2(C_f) + 0(\eta)$$

Within D_i, we have

$$\vec{V}_2 = \vec{V}_2' + \vec{V}_f = \vec{V}' = \vec{V}_E - \vec{V}_I$$

The last equation thus gives:

$$\vec{V}_R(M_{e_1})\cdot\overrightarrow{M_1M_1'} + \vec{V}_R(M_{e_2})\cdot\overrightarrow{M_2'M_2} + \Gamma_1(C_i) = \Gamma_2(C_f) + 0(\eta) \qquad (4.14)$$

where $\Gamma_1(C_i)$ is the circulation of $\vec{V}_1$ in C_i. Since

$$\Gamma_1(C_i) = \iint_{S_i} 2\vec{\Omega}_E\cdot\vec{\nu}\ dS_i$$

(S_i is an open surface whose edge is C_i and $\vec{\nu}$ is the unit vector normal to S_i), we see by taking for S_i a surface close to Σ_i that

$$\Gamma_1(C_i) = 0(\eta\cdot BB')$$

This shows that

$$\vec{V}_R(M_{e_1})\cdot\overrightarrow{M_1M_1'} - \vec{V}_R(M_{e_2})\cdot\overrightarrow{M_2M_2'} = \Gamma_2(C_f) + 0(\eta) \qquad (4.15)$$

Let η approach zero. This means either that the limiting positions B_1, B_2, B_f of M_1, M_2, M_f coincide with B or that the limiting positions

B_1', B_2', B_f' of M_1', M_2', M_f' coincide with B'. In fact, B is the limiting position of M_1, M_2, M_f and B' that of M_1', M_2', M_f'. To prove this, it suffices to consider the case when M_1, M_2, M_f are at B. If B_f' is not at B', it is somewhere between B' and M_f', and BB_f' is orthogonal to L' at B_f'. Since the free vortices move with the fluid, the fluid point P', which is at B_f' at the present time t, was at B' at a certain time t - dt. Consequently, the relative velocities at B_{f_1}' on Σ_{f_1} and at B_{f_2}' on Σ_{f_2}, which are orthogonal to L', are parallel to $B_f'B$. Similarly the relative velocity at B_{e_2}', which is orthogonal to L_2, is parallel to $B_2'B$ and the relative velocity at B_{e_1}', which is orthogonal to L_1' is parallel to $B_1'B$. Since BB' is infinitely small, $\vec{V}_R(B_{e_1})$ is parallel to BB_1' and $\vec{V}_R(B_{e_2})$ is parallel to BB_2' and both $\vec{V}_R(B_{f_2})$ and $\vec{V}_R(B_{f_1})$ are parallel to BB_f'. But $\vec{V}_R$ is continuous from Σ_{e_1} to Σ_{f_1} and from Σ_{e_2} to Σ_{f_2}. Consequently $\vec{V}_R(B_{e_1}) = \vec{V}_R(B_{f_1})$ and $\vec{V}_R(B_{e_2}) = \vec{V}_R(B_{f_2})$. It follows that BB_1' and BB_2' are parallel to BB_f' and that $\overrightarrow{BB_1'} = \overrightarrow{BB_2'} = \overrightarrow{BB_f'}$. By comparing Equations (4.13) and (4.15) and taking into account the continuity of $\vec{V}_R$ from Σ_{e_1} to Σ_{f_1} and from Σ_{e_2} to Σ_{f_2}, we could have obtained:

$$\vec{V}_R(B_{e_1})\cdot(\overrightarrow{BB_f'} - \overrightarrow{BB_1'}) - \vec{V}_R(B_{e_2})\cdot(\overrightarrow{BB_f'} - \overrightarrow{BB_2'}) = 0 \qquad (4.16)$$

But the above reasoning proves more since it gives

$$\overrightarrow{BB_f'} = \overrightarrow{BB_1'} = \overrightarrow{BB_2'} \qquad (4.17)$$

Furthermore we see that

$$\text{the trailing edge } \mathcal{B} \text{ is orthogonal to } \mathcal{L}_f, \mathcal{L}_1 \text{ and } \mathcal{L}_2 \tag{4.18}$$

Let us suppose that $\Gamma(C_f) > 0$. Then $\vec{V}_{R_1}$ is in the direction of $\overrightarrow{BB'}$ and $\vec{V}_{R_2}$, in the direction of $\overrightarrow{B'B}$. Consequently, the intensities $d\Gamma_1$, $d\Gamma_2$, and $d\Gamma_f$ of the vortex ribbons L_1, L_2, L_f bounded by $(\mathcal{L}_1, \mathcal{L}_1')$, $(\mathcal{L}_2, \mathcal{L}_2')$, and $(\mathcal{L}_f, \mathcal{L}_f')$ are connected to one another by

$$d\Gamma_f = d\Gamma_1 + d\Gamma_2 \tag{4.19}$$

This equation evidently holds if $\Gamma(C_f) < 0$. It follows that the two bound vortex filaments $\mathcal{L}_1$, $\mathcal{L}_2$ converge at B if $\vec{T}_f$ is in the outward direction with respect to the wing and that they start from $\mathcal{B}$ if $\vec{T}_f$ converges to $\mathcal{B}$. This also shows that L_1 and L_2 cannot be considered as included in the same vortex ribbon.

If the relative motion is steady, $|V_{R_1}|$ and $|V_{R_2}|$ are equal to each other and (see Figure 5)

$$d\Gamma_1 = d\Gamma_2 = \frac{1}{2}\, d\Gamma_f \quad \text{(steady relative motion)} \tag{4.20}$$

The properties indicated by Equations (4.18) and (4.19) do not seem to have been remarked on previously. They are the extension to the unsteady case for which results have been published by Roy.[7]

However, the above scheme implies a contradiction. For, if the relative velocity on Σ_{e_1} and Σ_{e_2} in the neighborhood of $\mathcal{B}$ were really parallel to $\mathcal{B}$, the fluid could not leave the surface Σ_e of the wing.

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[7]Roy, M., "Theorie des ailes sustentatrices et des hélices," Gauthier Villars, Paris (1934).

The bound vortex filaments L_1 on Σ_1 and L_2 on Σ_2 have probably such very high curvatures in the neighborhood of $\mathcal{B}$ that at a short distance from $\mathcal{B}$, the relative velocities at M_{e_1} and M_{e_2} are close to the relative velocities at M_{f_1} and M_{f_2}, respectively. But, clearly, the difficulty cannot be overcome within the framework of an almost inviscid fluid theory. Another aspect concerns the set of fluid points belonging to the free vortex filament L_f. This set increases continuously with time. This means that fluid points which were vortex-free at t may belong to L_f at t + dt. This contradicts the Helmholtz theorem. By assuming that the fluid is almost inviscid, one must necessarily accept contradictions somewhere. The contradiction here consists of (1) a certain discontinuity of the directions of $\vec{V}_R$ and (2) a discontinuity of its intensity at the origin and the end of each free vortex filament. In a real fluid, the boundary layer thickness is finite and the flow inside it is three-dimensional; on the outer side of the layer, the direction of the flow is not orthogonal to the vortex filament derived from the almost inviscid fluid theory and, furthermore, new fluid particles continuously enter the boundary layer.

The above contradiction does not prevent the almost inviscid fluid theory from being acceptable in regions other than those where the effects of viscosity are of considerable magnitude (as in the neighborhood of the trailing edge). In particular, L_1 and L_2 are unquestionably not parts of a unique bound vortex filament. Although L_2 is entirely located on the suction side, L_1 starts on the pressure side, reaches the leading edge, passes on to the suction side, comes back to the leading edge, and rejoins the trailing edge at the same point where L_2 ends. The parts Σ_1 and Σ_2 of Σ covered by L_1 and L_2 are separated by a line joining the two lateral edges of the wing and located on Σ_2. If there are two tip

vortices, this line belongs to the vortex filament which includes the tip vortices. The common ends of L_1 and L_2 are symmetrical to each other with respect to the transverse plane of symmetry of the wing. The line which starts from the midpoint of $\mathcal{B}$ and comes back to it is the frontier of a region Σ_3' covered with bound vortex filaments L_3 entirely located on Σ.

The wing surface Σ is thus divided into three regions Σ_1', Σ_2', and Σ_3' (see Figure 5). The lift follows from the fact that the number of arcs of vortex filaments located on the suction side Σ_2 includes all the filaments L_1, L_2, and L_3 whereas on the pressure side, there are only the vortex filaments L_1 and L_3.

THE GENERALIZED KUTTA CONDITION

As indicated earlier, this condition states that the relative velocity $\vec{V}_R$ is continuous from Σ_{e1} to Σ_{f1} and from Σ_{e2} to Σ_{f2}. Let us assume that the curvature of Σ along the trailing edge $\mathcal{B}$ is large but finite, i.e., that there exists at every point B of $\mathcal{B}$ a plane tangent to Σ. Let $\vec{\nu}_1$ (resp. $\vec{\nu}_2$) denote the unit vector with its origin at B, normal to $\mathcal{B}$ and tangent to Σ_1 (resp. Σ_2). Let B_{e_1} (resp. B_{e_2}) be a point infinitely close to B and belonging to Σ_{e_1} (resp. Σ_{e_2}). Two bound vortex filaments L_1 on Σ_1 and L_2 on Σ_2 end or begin at B. The components in the $\vec{\nu}_1$- direction of the velocities induced by the vortex distribution $\mathcal{D}_1$ and by the velocity potentials Φ_2' and Φ_f are, respectively,

$$0 \; ; \left(\vec{V}'(B_{i1})\cdot\vec{\nu}_1 + \frac{\partial\mu'}{\partial\nu_1}\right) + \left(-\vec{V}_f\cdot\vec{\nu}_1 + \frac{\partial\mu''}{\partial\nu_1}\right) \; ; \; \vec{V}_f\cdot\vec{\nu}_1$$

Here μ' and μ'' are the parts of μ_2' such that

$$\begin{cases} -\frac{1}{4\pi}\iint_{\Sigma} \mu'(P)\,\frac{\partial}{\partial n_p}\,\frac{1}{M_iP}\,d\Sigma(P) = \Phi'(M_i) \\ \\ -\frac{1}{4\pi}\iint_{\Sigma} \mu''(P)\,\frac{\partial}{\partial n_p}\,\frac{1}{M_iP}\,d\Sigma(P) = -\,\Phi_f(M_i) \end{cases}$$

The component in the $\vec{\nu}_1$-direction of the relative velocity at B_{e_1} is

$$\left[\vec{V}'(B_{i_1}) - \vec{V}_E\right]\cdot\vec{\nu}_1 + \left(\frac{\partial\mu'}{\partial\nu_1} + \frac{\partial\mu''}{\partial\nu_1}\right)_{B_1}$$

Since $\vec{V}_R(B_{e_1})$ must be parallel to $\mathcal{B}$, we must have

$$\left[\vec{V}'(B_{i_1}) - \vec{V}_E\right]\cdot\vec{\nu}_1 + \left(\frac{\partial\mu'}{\partial\nu_1} + \frac{\partial\mu''}{\partial\nu_1}\right)_{B_1} = 0 \qquad (4.21a)$$

Similarly, we must have

$$\left[\vec{V}'(B_{i_2}) - \vec{V}_E\right]\cdot\vec{\nu}_2 + \left(\frac{\partial\mu'}{\partial\nu_2} + \frac{\partial\mu''}{\partial\nu_2}\right)_{B_2} = 0 \qquad (4.21b)$$

But $\vec{V}'$ is continuous. Furthermore, $\vec{\nu}_1$ and $\vec{\nu}_2$ are in opposite directions. The above two conditions are thus equivalent. We may take for $\vec{\nu}$ one of the two vectors $\vec{\nu}_1$ and $\vec{\nu}_2$.

Let us come back to Equation (4.8). It may be written in the form

$$- \frac{1}{4\pi} \iint_{\Sigma} \mu_2'(P) \frac{\partial}{\partial n_p} \frac{1}{M_i P} d\Sigma(P) = \Phi'(M_i) - \Phi_f(M_i) = \psi'(M_i) \qquad (4.22)$$

Let $G(M, P)$ denote the resolvent kernel of this equation. We have

$$\mu_2'(M) = \iint_{\Sigma} G(M, P)\psi'(P) \, d\Sigma(P)$$

But Σ_f is unknown. To determine a first approximation of the solution, we have to make assumptions concerning the motion of Σ. For instance, if $\vec{V}_E$ is a constant and $\vec{\Omega}_E = 0$, we may consider Σ_f as generated by half-straight lines starting from $\mathcal{B}$ in the direction opposite to $\vec{V}_E$. Taking the x-axis in the direction of $\vec{V}_E$ and with σ a curvilinear abscissa on $\mathcal{B}$, we have

$$\Phi_f(P) = - \frac{1}{4\pi} \int_{\sigma_2}^{\sigma_1} d\sigma' \int_{x(\sigma')}^{-\infty} \mu_f(\sigma') \frac{\partial}{\partial n_Q} \frac{1}{PQ} \, dx(Q)$$

This can be written:

$$\Phi_f(P) = - \frac{1}{4\pi} \int_{\sigma_2}^{\sigma_1} d\sigma' \int_{x(\sigma')}^{-\infty} \mu_f(\sigma') H(\sigma', P, Q) \, dx(Q)$$

Hence at a point M belonging to Σ, we have

$$\mu''(M) = - \iint_{\Sigma} G(M, P)\Phi_f(P)\, d\Sigma(P)$$

$$= - \iint_{\Sigma} G(M, P)\, d\Sigma(P) \left[- \frac{1}{4\pi} \int_{\sigma}^{\sigma_2} d\sigma' \int_{x(\sigma')}^{-\infty} \mu_f(\sigma') H(\sigma', P, Q)\, dx(Q) \right] \tag{4.23}$$

The generalized Kutta condition therefore becomes:

$$\iint_{\Sigma} \frac{\partial G}{\partial \nu_B}(B, P)\, d\Sigma(P) \quad \left[- \frac{1}{4\pi} \int_{\sigma_2}^{\sigma_1} d\sigma' \int_{x(\sigma')}^{-\infty} \mu_f(\sigma') H(\sigma', P, Q)\, dx(Q) \right]$$

$$= - \left[\vec{V}'(B_i) - \vec{V}_E(B) \right] \vec{\nu}_B - \iint_{\Sigma} \frac{\partial G}{\partial \nu_B}(B, P) \Phi'(P_i)\, d\Sigma(P) \tag{4.24}$$

By interchanging the order of integrations, it may be written in the form

$$\frac{1}{4\pi} \int_{\sigma_2}^{\sigma_1} \mu_f(\sigma')\, d\sigma' \int_{x(\sigma')}^{-\infty} dx(Q) \iint_{\Sigma} \frac{\partial G}{\partial \nu_B}(B, P) H(\sigma', P, Q)\, d\Sigma(P)$$

$$= - \left[\vec{V}'(B_i) - \vec{V}_E(B) \right] \vec{\nu}_B - \iint_{\Sigma} \frac{\partial G}{\partial \nu_B}(B, P)\, \Phi'(P_i)\, d\Sigma(P) \tag{4.25}$$

The equation that determines μ_f on $\mathcal{B}$ is thus a Fredholm equation of the first kind. It is certainly very difficult to prove that this equation has a unique solution, but we may admit it for physical reasons. In practice, the use of the resolvent kernel of Equation (4.8) would

be difficult and if its numerical computation were necessary, other processes would have to be tried.

UNIFORM MOTION OF TRANSLATION OF A SUBMERGED BODY IN A VERTICAL PLANE

Now we consider a body with a longitudinal plane of symmetry zOx. This plane is vertical and is also symmetric with respect to the (x, y) plane which is horizontal when the squat is zero. The above symmetries are not needed by the theory, but they help to expound it. We assume that the body is in a uniform motion of translation. The velocity $\vec{V}_E$ of the origin of the moving axes O(x, y, z) is in the (z, x) plane. The angle of attack is $\delta = (Ox, \vec{V}_E)$. This is positive when the incident velocity $(- \vec{V}_E)$ has a positive z-component.

Experiments show that if $\delta \neq 0$, the system S^d of hydrodynamic forces exerted on the body includes a lift orthogonal to $\vec{V}_E$, even when the shape of the body differs considerably from that of a wing. This lift is due to the shedding of vortices. Otherwise, all the bound vortices would be closed, and according to the d'Alembert paradox, S^d would reduce to a couple. Various experiments (for instance, those carried out by Sears with an axisymmetric body) show that separation occurs at the afterbody and that the shedding of free vortices follows. (Figure 6)

More precisely, let us suppose that the forebody begins with a stem parallel to the z-axis and that the stern ends at a point S on the x-axis. (Figure 7) When $\delta = 0$, all of the relative streamlines start from the stem and end at S. We have a leading edge consisting of the stem, no trailing edge, and zero lift. When $\delta > 0$, the streamlines start not only from the stem but also from the lower part C_1 of the contour C formed by the intersection of the hull Σ with the (z, x)-plane.

The leading edge now consists of the stem and the arc C_1. Roughly speaking, the relative streamlines coming from the upper part of the stem seem to intersect those coming from the lower part of the stem and from

C_1 along a U-shaped line $\mathcal{B}$ consisting of an arc $B_oS'B'_o$ located on the afterbody. The ends B_o and B'_o of $\mathcal{B}$ are slightly downstream from the maximal transverse section of the body and symmetric to each other with respect to the $(\dot{z}, x)$-plane. The point S' is close to S.

The hull Σ is thus divided into three parts, Σ_1, Σ_2, Σ_3 ; Σ_2 is the upper part of the afterbody and its boundary is the line $\mathcal{B}$ and a half-ring $\mathcal{C}_2$ containing B_o and B'_o ; Σ_1 is the lower part of the afterbody and its boundary is the line $\mathcal{B}$ and a half-ring $\mathcal{C}_1$; the part Σ_3 is ahead of the ring $\mathcal{C}_1 + \mathcal{C}_2$ (see Figure 7).

There are three kinds of vortices. The vortex filaments L_3 are entirely bound. Their supports are rings on Σ_3. The vortex filaments L_1 come from infinity downstream and reach $\mathcal{B}$ at points B located on the portside (Figure 8); on Σ_1 their supports are half-rings starting from points B and ending at points B' which are symmetrical to points B and located on $\mathcal{B}$. They leave $\mathcal{B}$ from B' and extend back to infinity downstream. The vortex filaments L_2 are analogous to the L_1's but their bound parts are located on Σ_2. A free vortex L_f is the union of the free parts of one L_1 and of one L_2. If $d\Gamma_1$ and $d\Gamma_2$ are the intensities of L_1 and L_2, the intensity $d\Gamma_f$ of L_f is

$$d\Gamma_f = d\Gamma_1 + d\Gamma_2 \tag{4.26}$$

The scheme just defined is analogous to that for a wing of finite thickness except, of course, that the curvature of the hull along the line $\mathcal{B}$ is not necessarily high. If the fluid were really "almost inviscid," then not only the bound arcs of L_1 and L_2 but also the free arcs L_f should be orthogonal to $\mathcal{B}$. This would require a considerable curvature of the L_f's in the vicinity of $\mathcal{B}$, and, in fact, is impossible. Similarly, the relative velocity $\vec{V}_{R1}$ and $\vec{V}_{R2}$ on Σ_{e1} and Σ_{e2} cannot be in opposite directions along $\mathcal{B}$. The almost-inviscid fluid theory leads to contradictions

which are similar to those encountered in the case of the wings and perhaps is still open to criticism.

However, the only point of practical interest here is the assumption that free U-shaped vortices really do exist.

One may remark that the above scheme is not basically different from that considered by the slender body theory. To show this, consider a transverse section of the afterbody. (Figure 6). For the vertical force to be other than zero, we have to assume in the slender body theory that the flow is separated as seen in Figure 8, which implies the existence of a pair of longitudinal free vortices piercing the plane of the figure at T, T'. These free vortices necessarily originate on the body and their intensity grows in the negative x-direction. The slender body scheme thus seems to be derived from that proposed here by assuming that the length of the body is large enough for the free vortices originating ahead of the transverse section under consideration to have already wound up around the two "tip vortices."

My personal feeling is that because the real phenomenon is so complicated, the theory is able to provide us only with a usable frame to determine the general form of the equations. The number of the coefficients found in the equations should be as small as possible and their numerical values should be derived experimentally.

Another drawback is that the theory does not accurately tell us the position of the shedding line on the hull. But this should not entail the risk of large errors in the derivation of the system of forces exerted on the body. The generalized Kutta condition is expressed by the equation applicable to wings with a finite thickness.

DOUBLE MODEL IN AN OBLIQUE, UNIFORM MOTION OF TRANSLATION IN THE HORIZONTAL PLANE

Now consider a body whose shape is that of a double model relating to a surface ship (Figures 9 and 10). The (x,y)-plane is horizontal, the longitudinal plane is vertical. These two planes are planes of symmetry

for the double model. The velocity $\vec{V}_E(0)$ of the origin of the moving axes is in the horizontal plane and the drift angle $\delta = (\vec{V}_E, Ox)$ is positive. The lower half of the distribution of singularities kinematically equivalent to the body should be considered in the case of the so-called zero-Froude number approximation for a surface ship. It is known that this approximation is very rough for calculating the waves generated by a ship. But we will see later (Chapter 5) how to correct it. Thus the mathematical model described in the present section can be considered as a first approach to the maneuvering theory of a surface ship.

The z-axis is vertical upward. Let Σ_1 denote the portside $(y \geq 0)$ and Σ_2 the starboardside $(y \leqq 0)$. Let C be the contour intersected in the hull $\Sigma = \Sigma_1 + \Sigma_2$ by the (z, y)-plane, C_0 its lower part, C_1 its upper part, C_0 the arc EE_0S_0S, and C_1 the arc SS_1E_1E. The arc E_0S_0 is a horizontal straight line (keel). The arc E_1EE_0 of the double model is its stem, and the arc S_0SS_1, its sternpost. On C, the selected sense is that from Oz to Ox. The curvilinear abscissa is λ, and its origin is taken at E on the x-axis. We put

$$\lambda(E_0) = \lambda_0, \ \lambda(S_0) = \lambda_0', \ \lambda(S) = \pm \Lambda$$

$$\lambda(S_1) = \lambda_1' = - \lambda_0', \ \lambda(E_1) = \lambda_1 = - \lambda_0$$

The drift δ is small enough for no separation to occur. Experiments on surface ship models show that the relative streamlines are more inclined on the pressure side (starboard) than on the suction side (port). Thus, the two relative streamlines which reach the same point B_0 on the arc E_0S_0S do not have identical directions at B_0. This determines a shock and, as a consequence, a free vortex filament is shed from B_0. The origins on the stem of the two streamlines are not

at the same level, the origin of the streamline located on the pressure side being higher than that on the suction side.

Since we are de[illegible] with a double model, the same phenomenon occurs on the upper hal[illegible] the body. The arc playin[illegible] the role of trailing edge is the part [illegible]S_0SSE_1 of the contour C, an[illegible] the stem E_1EE_0 plays the role of leading edge. However, if the bottom is flat, then the after part of the shedding line tends to move toward port when δ increases. Beyond a critical value of δ, separation oc[illegible]rs. What follows does not apply in this case.

Let L_f be a free vortex filament which leaves arc SS_1E_1 from a point B_1. It goes to infinity downstream and comes back to the body at point B_0, the mirror image of B_1 with respect to the plane z = 0 and thus located on the arc SS_0E_0. By studying unsteady motions of the double model, we will see later that the bound part of the vortex filament L to which L_f belongs consists of a unique arc B_0B_1 necessarily located on the suction side. The phenomenon therefore differs from that for wings of finite thickness since, in the present case, L is not divided on Σ into two arcs L_{b_1}, L_{b_2} located on Σ_1, Σ_2, respectively.

If δ were zero, the bound vortex family would consist of vortex rings on transverse sections of the hull. The half-ring to port would be in the direction from B_0 to B_1, and the half-ring to starboard would be in the direction from B_1 to B_0. When δ is positive, then to port, we have a vortex filament of intensity $d\Gamma_1$ from B_0 to B_1, and to starboard, we have a vortex filament of intensity $d\Gamma_2$ from B_1 to B_0. The intensity $d\Gamma_1$ is greater than the intensity $d\Gamma_2$ since L is located to port and in the direction from B_0 to B_1. The two arcs B_0B_1 to port and B_1B_0 to starboard are no longer symmetric with each other. Let C_1' denote the arc B_0B_1 to port and C_2', the arc B_1B_0 to starboard. These two arcs depend on the abscissa λ of B_0 on E_0S_0S. Ahead of the arc $C_1'(\lambda_0) + C_2'(\lambda_0)$, the bound vortex filaments on Σ are vortex rings, but the parts of each ring located to port and starboard are not symmetric

to each other. We can consider that behind the ring $C_1'(\lambda_0) + C_2'(\lambda_0)$, we are dealing with two distributions of bound vortex filaments. One consists of vortex rings with intensity $d\Gamma_2(\lambda)$; the other consists of bound vortex filaments of intensity $d\Gamma_f(\lambda)$ located on the part $\Sigma_1(\lambda_0)$ of the portside, which is downstream from $C_1'(\lambda_0)$.

The vortex rings generate a velocity potential

$$\Phi_2'(M) = -\frac{1}{4\pi}\iint_{\Sigma}\mu_b'(P)\frac{\partial}{\partial n_p}\frac{1}{MP}\,d\Sigma(P) \tag{4.27}$$

On the other hand, let Σ_f be the free vortex sheet and Σ_{f_0}, Σ_{f_1}, Σ_f' the parts of Σ_f on which we find the free vortex filaments reaching the longitudinal contour between E_0 and S_0, those which leave C from E_1S_1, and those which leave C or reach C at points located on S_0SS_1. The vortex filament consisting of a free vortex filament L_f and of a bound vortex filament L_b'' located on the part $\Sigma_1(\lambda_0)$ of the portside is equivalent to a normal dipole distribution on the part of $\Sigma_f + \Sigma_1$ downstream from its contour.

It follows that the part of vortex distribution $\mathcal{D}_2'$ due to all the vortex filaments $L_b'' + L_f$ is equivalent to a normal doublet distribution on $\Sigma_f + \Sigma_1$:

$$\Phi_2''(M) = -\frac{1}{4\pi}\iint_{\Sigma_1}\mu_b''(P)\frac{\partial}{\partial n_p}\frac{1}{MP}\,d\Sigma_1(P) - \frac{1}{4\pi}\iint_{\Sigma_f}\mu_f(P)\frac{\partial}{\partial n_p}\frac{1}{MP}\,d\Sigma_f(P) \tag{4.28}$$

The sum $\Phi_2' + \Phi_2''$ may be written in the form

$\Phi_2(M) + \Phi_f(M)$, with

$$\Phi_2(M) = -\frac{1}{4\pi}\iint_\Sigma \mu'(P)\frac{\partial}{\partial n_p}\frac{1}{MP}\, d\Sigma(P) \tag{4.29}$$

$$\Phi_f(M) = -\frac{1}{4\pi}\iint_{\Sigma_f} \mu_f(P)\frac{\partial}{\partial n_p}\frac{1}{MP}\, d\Sigma_f(P)$$

The function μ' defined on Σ has three different determinations according to whether P is located on $\Sigma_1(\lambda_0)$, on $\Sigma_2(\lambda_0)$, or on the part Σ_3 of Σ ahead of $C_1'(\lambda_0) + C_2'(\lambda_0)$. It is determined by the condition that

$$\Phi_2(M) = -\Phi_f(M) + V_E(0)\ (x \cos\delta - y \sin\delta) \quad \text{within } D_i \tag{4.30}$$

This gives:

$$-\frac{1}{2}\mu'(M) - \frac{1}{4\pi}\iint_\Sigma \mu'(P)\frac{\partial}{\partial n_p}\frac{1}{MP}\, d\Sigma(P) =$$

$$- \Phi_f(M_i) + V_E(0)\ [x \cos\delta - y \sin\delta] \quad \text{for } M \in \Sigma \tag{4.31}$$

The three determinations of μ' are distinct from one another because of the antisymmetry between the two sides of Σ (due to the antisymmetry of Σ_f with respect to the double model) and also because no free vortex filament is shed from the arc E_0EE_1. On Σ_f, μ_f is a constant along the free vortex filament L_f. Hence, Φ_f depends only

on the function $\frac{d\Gamma_f}{d\lambda}$ on the arc $E_0S_0SS_1E_1$ of C. Let us suppose that this function is known. Then μ' is determined by Equation (4.31). The arcs $C_1(\lambda)$ and $C_2(\lambda)$ are the lines μ' = constant and the relative streamlines on Σ are the curves $\mathcal{C}$ orthogonal to these lines. To determine the flow entirely, a complementary condition is needed. This is the generalized Kutta condition.

THE GENERALIZED KUTTA CONDITION FOR BODIES IN AN OBLIQUE, UNIFORM TRANSLATION IN THE HORIZONTAL PLANE

Let B_0 be a point on the arc E_0S_0S of the longitudinal contour C of the double model and let Π_0 be the plane tangent to Σ at B_0 (Figure 11). Let $B_0(y_0', \lambda', z_0')$ denote a right-handed system of axes, $B_0\lambda'$ being tangent to C in the direction of increasing λ's. The axis B_0y_0' is in the positive y-direction. The axis B_0z_0' is not necessarily vertical upward but its direction coincides with the inward normal $\vec{n}$ at B_0. We consider the right-handed systems of unit vectors $(\vec{n}, \vec{\theta}_1, \vec{\tau}_1)$ and $(\vec{n}, \vec{\theta}_2, \vec{\tau}_2)$ for the port and starboard sides, respectively. We recall that the directions of $\vec{V}_{R1}$, $\vec{V}_{R2}$

are opposite to $\vec{\theta}_1$ and $\vec{\theta}_2$, respectively. In the above two sketches, Figure 11, we have taken into account the fact that the angle $|(\vec{V}_{R1}, B_0\lambda')|$ is less than the angle $|(\vec{V}_{R2}, B_0\lambda')|$.

In the neighborhood of B_0, $z_0 = O(y_0'^2 + \lambda'^2)$. Hence, Γ_1 and Γ_2 can be considered in that region as functions of (y_0', λ'). Since Γ_1 is constant along $C_1'(\lambda)$ and $\Gamma_2(\lambda)$ is constant along $C_2'(\lambda)$, the equations of the tangents at B_0 to $C_1'(\lambda)$ and $C_2'(\lambda)$ are as follows:

$$\begin{aligned} \frac{\partial\Gamma_1}{\partial y_0'} y_0' + \frac{\partial\Gamma_1}{\partial\lambda'} \lambda' = 0 \quad , \quad y_0' \geq 0 \\ \frac{\partial\Gamma_2}{\partial y_0'} y_0' + \frac{\partial\Gamma_2}{\partial\lambda'} \lambda' = 0 \quad , \quad y_0' \leq 0 \end{aligned} \tag{4.32}$$

The equations of the tangents to the streamlines $\mathcal{C}_1$ and $\mathcal{C}_2$ ending at B_0 are

$$\begin{aligned} \frac{\partial\Gamma_1}{\partial\lambda'} y_0' - \frac{\partial\Gamma_1}{\partial y_0'} \lambda' = 0 \quad , \quad (y_0' \geq 0) \\ \frac{\partial\Gamma_2}{\partial\lambda'} y_0' - \frac{\partial\Gamma_2}{\partial y_0'} \lambda' = 0 \quad , \quad (y_0' \leq 0) \end{aligned} \tag{4.33}$$

Note that

$$\frac{\partial\Gamma_1}{\partial\lambda'} \cdot \frac{\partial\Gamma_1}{\partial y_0'} < 0 \quad , \quad \frac{\partial\Gamma_2}{\partial\lambda'} \cdot \frac{\partial\Gamma_2}{\partial y_0'} > 0$$

Let M_1, M_2 denote the intersections of C_1 with $C_1'(\lambda - d\lambda)$ and of C_2 with $C_2'(\lambda - d\lambda)$. We have

$$d\sigma_1 = M_1 B_0 \quad , \quad V_{R_1} = \frac{d\Gamma_1}{d\sigma_1}$$

This gives:

$$V_{R_1}(B_0) = \frac{d\Gamma_1}{d\lambda'} \frac{1}{\cos\,(B_0 y_0' \, , \, \tau_1')} =$$

$$\frac{d\Gamma_1}{d\lambda'} \frac{\left[\left(\frac{\partial\Gamma_1}{\partial y_0'}\right)^2 + \left(\frac{\partial\Gamma_1}{\partial\lambda'}\right)^2\right]^{\frac{1}{2}}}{\frac{\partial\Gamma_1}{\partial\lambda'}} \quad , \text{ for } y_0 = 0$$

But $d\Gamma_1 = \frac{\partial\Gamma_1}{\partial\lambda'} \, d\lambda'$ on $E_0 S_0 S$. Consequently

$$V_{R_1}(B_0) = \sqrt{\left(\frac{\partial\Gamma_1}{\partial y_0'}\right)^2 + \left(\frac{\partial\Gamma_1}{\partial\lambda'}\right)^2} \tag{4.34a}$$

and similarly

$$V_{R_2}(B_0) = \sqrt{\left(\frac{\partial\Gamma_2}{\partial y_0'}\right)^2 + \left(\frac{\partial\Gamma_2}{\partial\lambda'}\right)^2} \tag{4.34b}$$

The generalized Kutta condition states that the pressure on the hull is continuous through E_0S_0S and therefore that

$$V_{R_1}^2\,(B_0) = V_{R_2}^2\,(B_0) \tag{4.35}$$

This gives:

$$\frac{d\Gamma_f}{d\lambda} = \frac{d\Gamma_1}{d\lambda'} - \frac{\partial\Gamma_2}{d\lambda'} = \left.\frac{\left(\frac{\partial\Gamma_1}{\partial\lambda'}\right)^2 - \left(\frac{\partial\Gamma_2}{\partial\lambda'}\right)^2}{\frac{\partial\Gamma_1}{\partial\lambda'} + \frac{\partial\Gamma_2}{\partial\lambda'}}\right|_{y_0'=0} = -\left.\frac{\left(\frac{\partial\Gamma_1}{\partial y_0'}\right)^2 - \left(\frac{\partial\Gamma_2}{\partial y_0'}\right)^2}{\frac{\partial\Gamma_f}{\partial\lambda} + 2\,\frac{\partial\Gamma_2}{\partial\lambda'}}\right|_{y_0'=0} \tag{4.36}$$

For $\frac{d\Gamma_f}{d\lambda}$ to be other than negligibly small, the slope of $C_1'(\lambda)$ with respect to B_0y_0' and that of $C_2'(\lambda)$ with respect to $B_0\lambda'$ must be relatively small. Thus is seems that

$$\frac{\partial\Gamma_1}{\partial\lambda'} >> -\frac{\partial\Gamma_1}{\partial y_0'}\ , \quad \frac{\partial\Gamma_2}{\partial y_0'} >> \frac{\partial\Gamma_2}{\partial\lambda'} \tag{4.37}$$

This gives

$$\frac{d\Gamma_f}{d\lambda} \cong \frac{\partial\Gamma_2}{\partial y_0'} \tag{4.38}$$

It is difficult to get more information unless numerical computation is attempted.

FINAL FORMULAS FOR BODIES IN AN OBLIQUE,
UNIFORM TRANSLATION IN THE HORIZONTAL PLANE

Let P be a point on Σ and let P_i, P_e be the points defined by $\overrightarrow{PP_i} = \vec{n}_p(0+)$ and $\overrightarrow{PP_e} = -\vec{n}_p(0+)$, respectively. Let $d\sigma$ be the element of arc of the relative streamline which passes through P_e on Σ_e or that of the relative streamline determined on Σ by the vortex distribution. We have $\frac{\partial \mu'}{\partial \sigma} = \frac{\partial \Gamma}{\partial \sigma}$ where μ' is the density of the normal doublet distribution in (4.31) and $\frac{\partial \Gamma}{\partial \sigma}\vec{\tau}$ is the intensity of the vortex $\vec{T}$ (P) on Σ. Thus we have

$$\mu'(P) = \Gamma(P) = \Phi_2(P_e) - \Phi_2(P_i) \tag{4.39}$$

Here Γ is defined up to an additive constant; it increases from the stem to the stern post, with $\Gamma=\Gamma_1$, Γ_2 or Γ_3 according to whether P is on Σ_1, Σ_2, or Σ_3. On $\Sigma_f = \Sigma_{f1} + \Sigma_{f0} + \Sigma'_f$, we have at any point:

$$\mu_f(P_f) = \mu_f(B_0) \tag{4.40}$$

where B_0 is the point located on E_0SS_0 which belongs to the free vortex filament containing P_f.

The unit vector normal to Σ_f at P_f is selected so that it has a direction coherent with that of $\vec{n}$ on Σ. Thus it is in the upward direction on Σ_{f1} and in the downward direction on Σ_{f0}; on Σ'_f, it is in the direction from port to starboard. The points P_f^- and P_f^+ are defined by

$$\overrightarrow{P_fP_f^+} = \vec{n}_f(0+) \quad , \quad \overrightarrow{P_fP_f^-} = -\vec{n}_f(0+)$$

We easily see that

$$\mu_f(P) = \Phi_f(P^-) - \Phi_f(P^+) = \Gamma_f(B_0) \tag{4.41}$$

Finally, we may write:

$$\Phi(M) = \Phi_2(M) + \Phi_f(M) \text{ with} \tag{4.42}$$

$$\left\{\begin{aligned} \Phi_2(M) &= \frac{-1}{4\pi} \iint_{\Sigma} \Gamma(P) \frac{\partial}{\partial n_P} \frac{1}{MP} \, d\Sigma(P) \\ \Phi_f(M) &= -\frac{1}{4\pi} \iint_{\Sigma_{f_1} + \Sigma_{f_0} + \Sigma'_f} \Gamma_f(B_0) \frac{\partial}{\partial n_P} \frac{1}{MP} \, d\Sigma(P) \end{aligned}\right. \tag{4.43}$$

Γ is determined by the condition:

$$\left\{\begin{aligned} &-\frac{1}{2}\Gamma(M) - \frac{1}{4\pi} \iint_{\Sigma} \Gamma(P) \frac{\partial}{\partial n_P} \frac{1}{MP} \, d\Sigma(P) = -\Phi_f(M) + V_E(0)\,[x \cos\delta - y \sin\delta] \quad M \varepsilon \Sigma, \\ &\frac{d}{d\lambda}\Gamma_f(B_0) \cong \left(\frac{\partial \Gamma_2}{\partial y_0}\right) B_0 \quad , \quad \lambda(B_0) \,\varepsilon\, [\lambda_0, \Lambda] \end{aligned}\right. \tag{4.44}$$

Furthermore, we have in the first iteration:

$$\overrightarrow{P_f B_0} \cong [x(B_0) - x(P_f)] \, \vec{i}(\vec{V}_E) \quad \text{on } \Sigma_f \tag{4.45}$$

where the difference in square brackets indicates the difference between the abscissas of B_0 and P measured in the direction of $\vec{V}_E$.

The second formula, Equation (4.44) is the reproduction of Equation (4.38). Since the latter is an approximate equation, used only because at the first stage Γ is unknown, it would be convenient after this first computation, to check whether it is necessary to replace (4.38) by (4.36) (the values of Γ_1 and Γ_2 in (4.36) are those given by the first iteration). This would lead to the second iteration.

CHAPTER 5: PROBLEMS RELATING TO THIN SHIPS AND TO FREE SURFACE EFFECTS

It happens very often in ship hydrodynamics that drastic simplifications of a problem yield interim solutions which later become useful bases for a more refined treatment of the real case. This has been so for thin ship theory in the case of wave resistance. For this reason, the present chapter attacks the problem of the maneuverability of a very thin ship and examines how to take into account the free surface effect when the boundary condition on the free surface can be linearized.

TENTATIVE THEORY FOR INFINITELY THIN SHIPS IN AN OBLIQUE, UNIFORM TRANSLATION (STEADY CASE)

General Comments

In a dissertation prepared under my direction about 20 years ago, Casal outlined a theory of small aspect ratio wings for its applicability to ship maneuverability. His results are of interest from many points of view. The picture he gave of the hydrodynamic forces exerted on a maneuvering ship is, in general, in rather good qualitative agreement with experiments. However, his assumptions were rather crude, and one wonders whether their refinement could lead to numerical calculations in a relatively simple manner.

As mentioned previously, the hypothesis that the ship is very thin has not been discarded, but assumptions concerning the bound vortex distribution on the longitudinal plane of symmetry of the hull have been eliminated. This was done in order for the boundary condition on the hull to be satisfied everywhere and not merely on the longitudinal axis of the longitudinal plane. Of course, no real ship is thin. But, the addition of a source distribution on the longitudinal plane might enable the treatment of bodies with shapes relatively close to those of real ships. Indeed, the very thin wing gives rise to difficulties for the leading edge. (This had been mentioned at the beginning of Chapter 4.) But thanks to the additional source distribution, it would probably be possible to remove them.

It will be seen below that the greatest difficulty originates in combining the thinness of the model with the very small aspect ratio of a real ship form. The latter is sensibly less than unity but, in the case of

wings, it is usually much greater than unity. The shedding line begins at the junction of the stem with a keel; thus, the phenomena occurring in the neighborhood of the lateral edges of a normal wing are considerably magnified. The numerical calculation of the solution would probably be more intricate than that of the solution sketched in Chapter 4 for real ship forms. For this reason, I was almost at the point of withdrawing the subject of the very thin ship from the present lectures. I retained it, however, because it may throw light on the nature of the problem.

Notations and Assumptions

The double model of a thin ship is reduced to a vertical flat plate in the (z, x)-plane. The horizontal plane (Ox, Oy) is that of the waterline of the surface ship. The axes O(x, y, z) move with the body. Also, a second system of axes O (x_1, y_1, z) is introduced for convenience. The planes (Ox_1, Oy_1) and (Ox, Oy) coincide; the x_1-axis is in the direction of the constant velocity $\vec{V}_E$ of the common origin of the two systems of axes.

The contour C of the plate consists of the stem EE_0 of the surface ship, of its keel E_0S_0, of its stern post S_0S, and of the other three arcs EE_1, E_1S_1, S_1S which are the mirror images of the former with respect to the (x,y)-plane. The x-axis is in the direction from S to E. The z-axis is vertical upward. It intersects the keel in the midpoint of the segment E_0S_0. Let 2ℓ be the length of this segment. The keel E_0S_0 is horizontal. The drift angle

$$\delta = (Ox_1, Ox)$$

is supposed to be positive and small enough for no separation to occur on the real ship. No free vortex is shed from the stem E_1EE_0 of the double model. The shedding line is the arc $E_0S_0SS_1E_1$. The supports of the free vortices are assumed to be in the negative x_1-direction. A free vortex filament L_δ consists of two arcs, $L_{\delta 1}$ and $L_{\delta 0}$ or $L'_{\delta 1}$ and $L'_{\delta 0}$. $L_{\delta 1}$ is in the plane z = H; it leaves the arc S_1E_1 at a certain point B_1; $L_{\delta 0}$ is in the plane z = -H and reaches the contour C at a point B_0, mirror image of B_1. $L_{\delta 1}$ generates the part $\Sigma_{\delta 1}$ of the free vortex sheet Σ_δ, while L_δ generates the part $\Sigma_{\delta 0}$. The free vortex filament $L'_{\delta 1}$ leaves the upper half of the stern

post at a certain point B_1', and $L_{\delta 0}$ ends on the lower half of the stern post at B_0', mirror image of B_1'. $L'_{\delta 1}$ and $L'_{\delta 0}$ generate the part Σ'_δ of Σ_δ. A free vortex filament L_δ (or L'_δ) is closed by a bound vortex filament L_b which joins B_0 to B_1 (or B_0' to B_1').

The contour C is oriented in the positive direction (from Oz to Ox). Let ds denote an element of arc of C_1 and let $\vec{\nu}$ be the unit vector normal to C in the outward direction. Hence, if ds>0, $(\overrightarrow{ds}, \vec{\nu}) = \frac{-\pi}{2}$. Let $x=-X(z)$ be the equation of the stern post in the plane y = 0. Let $\vec{\omega}_b = \frac{\vec{T}_b}{\varepsilon}$ be the bound vortex at a point M of the area Σ of the longitudinal plane of the hull. As ε is a constant length equal to 0+, $\vec{T}_b$ is divergenceless on Σ. Thus, there exists a function ψ of the variables (z, x) defined on Σ such that

$$\vec{T}_b(M) = \mathrm{curl}\,(\psi(M)\vec{i}y) \tag{5.1}$$

where $\vec{i}y$ is the unit vector in the positive y-direction. The components of $\vec{T}_b$ on the axes O(x, y, z) are denoted

$$\xi = -\frac{\partial\psi}{\partial z}, \ \eta = 0 \ , \ \zeta = \frac{\partial\psi}{\partial x} \tag{5.2}$$

The flux of $\vec{T}_b$ through an arc ds of C in the <u>outward</u> direction is

$$d\Gamma = \vec{T}_b \cdot \vec{\nu}\, ds = \frac{\partial\psi}{\partial s}\, ds \tag{5.3}$$

This quantity is the common intensity of the free vortex filament L_δ and of the bound vortex filament L_b intersecting the arc E_0S_0S at B_0 or B_0' and the arc E_1S_1S at B_1 or B_1'. The function ψ is an <u>odd</u> function of z and $\partial\psi/\partial z$ is <u>even</u>. According to the above assumption concerning the stem, ψ is constant on E_1EE_0.

For convenience, we note

$$\begin{aligned} x_o &= \text{the abscissa of } B_0 \text{ on } E_0S_0 \\ -X_o(z_o) &= \text{the abscissa of } B_0' \text{ on } S_0S \end{aligned} \tag{5.4}$$

The velocity $\vec{V}$ induced at M by the total vortex distribution is the sum of the contributions $\vec{V}(M/\Sigma_{\delta 0})$ from $\Sigma_{\delta 0}$, $\vec{V}(M/\Sigma_{\delta 1})$ from $\Sigma_{\delta 1}$, $\vec{V}(M/\Sigma'_{\delta})$ from Σ'_{δ}, and $\vec{V}(M/\Sigma)$ from Σ. It will be calculated by using the Biot and Savart formula. Before performing this calculation, let us remark that $\vec{T}_b$ is not tangent to C (except on E_1EE_0). Therefore L_{δ} cannot be in the direction parallel to $\vec{V}_E$ on E_0S_0, E_1S_1, and S_0SS_1. This contradicts our assumption concerning the geometry of the free vortex filaments. The assumption will be slightly modified when dealing with the generalized Kutta condition.

Velocity Induced by the Free and the Bound Vortex Sheets

It is easier to use the system of axes $O(x_1, y_1, z)$. Let M' be at (x_1', y_1', z_1') or (x', y', z') on Σ_{δ}. We have

$$u_1(M/\Sigma_{\delta} + \Sigma_{\delta 1} + \Sigma'_{\delta}) = 0 \tag{5.5}$$

$$\left\{\begin{aligned}
v_1(M/\Sigma_{\delta 0}) &= \frac{-1}{4\pi}\int_{-\ell}^{\ell}\frac{\partial\psi}{\partial x_0}\,dx_0\int_{-\infty}^{x_0\cos\delta}\frac{-H-z_1}{|MM'|^3}\,dx_1';\ (y_1'=x_0\sin\delta,\ z_1'=-H,\ z_0=-H)\\
v_1(M/\Sigma_{\delta 1}) &= \frac{-1}{4\pi}\int_{-\ell}^{\ell}\frac{\partial\psi}{\partial x_0}\,dx_0\int_{-\infty}^{x_0\cos\delta}\frac{H-z_1}{|MM'|^3}\,dx_1';\ (y_1'=x_0\sin\delta,\ z_1'=H,\ z_0=H)\\
v_1(M/\Sigma'_{\delta}) &= \frac{-1}{4\pi}\int_{-H}^{H}\left(\frac{d\psi}{dz_0}\right)dz_0\int_{-\infty}^{-X_0\cos\delta}\frac{z_1'-z_1}{|MM'|^3}\,dx_1';\ (y_1'=-X_0\cos\delta,\ z_1'=z_0,\ x_0=-X_0)
\end{aligned}\right. \tag{5.6}$$

$$
\begin{cases}
w_1(M/\Sigma_{\delta 0}) = \dfrac{1}{4\pi}\displaystyle\int_{-\ell}^{\ell} \dfrac{\partial\psi}{dx_0}\, dx_0 \int_{-\infty}^{x_0\cos\delta} \dfrac{x_0 \sin\delta - y_1}{|MM'|^3}\, dx_1' \\[2ex]
w_1(M/\Sigma_{\delta 1}) = \dfrac{1}{4\pi}\displaystyle\int_{-\ell}^{\ell} \dfrac{\partial\psi}{\partial x_0}\, dx_0 \int_{-\infty}^{x_0\cos\delta} \dfrac{x_0 \sin\delta - y_1}{|MM'|^3}\, dx_1' \\[2ex]
w_1(M/\Sigma_{\delta}') = \dfrac{1}{4\pi}\displaystyle\int_{-H}^{H} \left(\dfrac{d\psi}{dx_0}\right) dz_0 \int_{-\infty}^{-X_0\cos\delta} \dfrac{-X_0 \cos\delta - y_1}{|MM'|^3}\, dx_1'
\end{cases}
\tag{5.7}
$$

Behavior of $\underline{v_1(M/\Sigma_\delta)}$. Let us suppose that

$$
\left|\frac{\partial\psi}{\partial x_0}\right| < \infty \text{ on } S_0E_0 \text{ and } \left|\frac{\partial\psi}{\partial z_0}\right| < \infty \text{ on } S_0S \tag{5.8}
$$

Then v_1 is bounded on Σ and on the contour of Σ (except at S_0 and S_1 if $\frac{\partial\psi}{\partial z_0} \neq 0$ at these two points).

Behavior of $\underline{w_1(M/\Sigma_\delta)}$. w_1 is also bounded on Σ. But on S_0E_0 and S_1E_1, $|w|=\infty$ because of the assumption that L_δ is parallel to $\vec{V}_E$ at its two ends.*

Velocity Induced by the Bound Vortex Sheet. By using the axes O(x, y, z), we have

$$
\begin{cases}
u(M/\Sigma) = \dfrac{1}{4\pi} \displaystyle\iint_\Sigma \zeta(M') \dfrac{\partial}{\partial y} \dfrac{1}{MM'}\, d\Sigma(M') \\[2ex]
v(M/\Sigma) = \dfrac{1}{4\pi} \displaystyle\iint_\Sigma \left[\xi(M') \dfrac{\partial}{\partial z} - \zeta(M') \dfrac{\partial}{\partial x}\right] \dfrac{1}{MM'}\, d\Sigma(M') \\[2ex]
w(M/\Sigma) = -\dfrac{1}{4\pi} \displaystyle\iint_\Sigma \xi(M') \dfrac{\partial}{\partial y} \dfrac{1}{MM'}\, d\Sigma(M')
\end{cases}
\tag{5.9}
$$

*See the section dealing with boundary conditions on the hull and the one on the generalized Kutta condition below.

The components u and w may be regarded as the normal derivatives on Σ of potentials due to sources with respective densities $-\zeta(u')$ and $\xi(M')$. Let Σ^+ and Σ^- be the surfaces derived from Σ by translation $\pm$ 0 in the direction of Oy. Then

$$u^+(M/\Sigma) - u^-(M/\Sigma) = -\zeta(M)$$

$$w^+(M/\Sigma) - w^-(M/\Sigma) = +\xi(M)$$

As $\zeta > 0$,

$$u^+(M/\Sigma) - u^-(M/\Sigma) > 0$$

The absolute value of the x-component of the relative velocity is larger on the suction side (port) than on the pressure side (starboard). Furthermore, the z-component of $\vec{V}$ (M/Σ), which is negative for $z = -H$, is less in absolute value to port than to starboard if $\xi > 0$ on S_0E_0.

This shows that our mathematical model agrees with the cause of shedding along S_0E_0 (and S_1E_1) and that the pressure is effectively smaller to port than to starboard. The y-component of $\vec{V}$ (M/Σ) is obviously continuous through Σ. It may be written in the form

$$v(M/\Sigma) = \frac{1}{4\pi} \iint_{\Sigma} \left(\frac{\partial\psi}{\partial z'} \frac{\partial}{\partial z'} \frac{1}{MM'} + \frac{\partial\psi}{\partial x'} \frac{\partial}{\partial x'} \frac{1}{MM'} \right) d\Sigma(M')$$

$$= \frac{1}{4\pi} \iint_{\Sigma} \left[\frac{\partial}{\partial z'} \left(\frac{1}{MM'} \frac{\partial\psi}{\partial z'} \right) + \frac{\partial}{\partial x'} \left(\frac{1}{MM'} \frac{\partial\psi}{\partial x'} \right) - \frac{1}{MM'} \Delta\psi(M') \right] d\Sigma(M')$$

that is,

$$v(M/\Sigma) = \frac{1}{4\pi} \int_{C^+} \frac{1}{MM'} \left(\frac{\partial\psi}{\partial z'} dx' - \frac{\partial\psi}{\partial x'} dz' \right) - \frac{1}{4\pi} \iint_{\Sigma} \frac{1}{MM'} \Delta\psi(M') \, d\Sigma(M') \qquad (5.10)$$

In this formula, $\Delta = \dfrac{\partial^2}{\partial z'^2} + \dfrac{\partial^2}{\partial x'^2}$

The Boundary Condition on the Hull

This condition is

$$\vec{i}y \cdot \vec{V}\,(M) = -\,C \sin\delta \text{ for } y = 0 \qquad (5.11)$$

or

$$v_1(M/\Sigma_\delta) \cos\delta + v(M/\Sigma) = -\,C \sin\delta \;\underline{\text{on } \Sigma} \qquad (5.12)$$

According to the previous section, this gives:

$$\frac{1}{4\pi} \iint_{\Sigma} \frac{1}{MM'} \Delta\psi(M') \, d\Sigma(M) = c \sin\delta + \frac{1}{4\pi} \int_{C^+} F(M,M') \frac{\partial\psi}{\partial s}(M') \, ds(M')$$

$$+ \frac{1}{4\pi} \int_{C^+} \left(\frac{\partial\psi}{\partial\nu}\right)_{M'} \frac{1}{MM'} \, ds(M'), \quad (M \epsilon \Sigma) \qquad (5.13)$$

with

$$\frac{\partial\psi}{\partial s} = 0 \quad \text{on} \quad E_1EE_0 \qquad (5.14)$$

F(M,M') on the right-hand side of Equation (5.13) is a known function of M and M', which also depends on the drift δ.

The Generalized Kutta Condition for an Infinitely Thin Double Model

For the sake of brevity, let us put

$$C_1 = E_0S_0SS_1E_1 = C - E_1EE_0 \tag{5.15}$$

Let M be a point inside the bound vortex sheet and M^-, M^+ the points derived from M by $\overrightarrow{MM^+} = \vec{i}y(0+)$, $\overrightarrow{MM^-} = -\vec{i}y(0+)$. The Kutta condition states that

$$p(M^+) = p(M^-) \quad \text{if} \quad M \in C_1 \tag{5.16}$$

It is equivalent to the condition that

$$\vec{V}_R(M) \cdot [\vec{V}_R(M^+) - \vec{V}_R(M^-)] \equiv 0 \text{ on } C_1 \tag{5.17}$$

Since $w(M/\Sigma) = 0$, $u(M/\Sigma) = 0$, we have

$$\vec{V}_R(M) = w(M/\Sigma_\delta)\, \vec{i}_z + [u(M/\Sigma_\delta) - c \cos \delta]\, \vec{i}_x$$

Hence (5.17) may be written in the form:

$$\frac{\partial}{\partial z} \psi(M)\, w(M/\Sigma_\delta) + \frac{\partial}{\partial x} \psi(M)\, [u(M/\Sigma_\delta) - c \cos \delta] = 0 \text{ on } C_1 \tag{5.18}$$

We have seen that the assumption that the free vortices reach C or leave C in a direction parallel to $\vec{V}_E$ entails

$$|w(M/\Sigma_\delta)| = \infty \text{ on } S_1E_1 \text{ and } E_0S_0$$

Thus, this assumption as to be modified.

We now assume that if M_δ is inside the free vortex sheet, then $|\vec{V}(M_\delta)| = |\vec{V}(M)|$ and $\vec{V}_R(M_\delta)$ is parallel to $\vec{V}_E$ (provided that $|MM_\delta| > 0$). This gives

$$\vec{V}_R(M_\delta) = [u(M/\Sigma_\delta) - c \cos \delta]\, \vec{i}_x + w(M/\Sigma_\delta)\, \vec{i}_y$$

Inasmuch as this vector is parallel to V_E, its y_1-component is zero and

$$[u(M/\Sigma_\delta) - c \cos\alpha] \sin\delta + w(M/\Sigma_\delta) \cos\delta = 0, \ |MM_\delta| > 0 \qquad (5.19)$$

By substituting this into Equation (5.18), we obtain

$$\begin{cases} \frac{-\partial}{\partial z}\psi(M) \tan\delta = \frac{\partial}{\partial x}\psi(M) \text{ on } S_0E_0, \\ \frac{\partial}{\partial z}\psi(M) \tan\delta = \frac{\partial}{\partial x}\psi(M) \text{ on } S_1E_1 \end{cases} \qquad (5.20)$$

At S, $w(M/\Sigma_\delta) = 0$ for reason of symmetry, and $w(M/\Sigma_\delta)$ is probably small on S_0S_1. We thus assume that $w(M/\Sigma_\delta) = 0$ on S_0S_1. This gives, after Equation (5.18):

$$\frac{\partial}{\partial x}\psi(M) = 0 \quad \text{on} \quad S_0SS_1 \qquad (5.21)$$

and we recall that

$$\frac{\partial}{\partial x}\psi(M) = 0 \quad \text{on} \quad E_1EE_0 \qquad (5.22)$$

Formula (5.20) shows that the z-component of $\vec{T}_b$ is much less than its x-component in the vicinity of E_0S_0 and S_1E_1. Hence the curvature of the vortex filaments is generally high on the plate.

The Integral Equation of the Problem

The general solution of the equation

$$\Delta\psi = f(M) \text{ inside } \Sigma \qquad (5.23)$$

is $\psi = \psi_1 + \psi_0$ with

$$\psi_1 = \frac{1}{2\pi} \iint_\Sigma \log |MM'| \; f(M') \; d\Sigma(M'),$$

$$\Delta\psi_0(M') \equiv 0 \quad \text{inside } \Sigma \tag{5.24}$$

Our purpose is to show that Equation (5.13) allows ψ to be expressed in terms of the harmonic function ψ_0, and that the generalized Kutta condition determines ψ_0.

To that end, let us consider the operators F_1, F_1^{-1} which define the Fourier transform of a given function inside Σ and its inverse. We have

$$\begin{cases} \tilde{\varphi}(\theta,k) = F_1\{\varphi\} = \dfrac{1}{2\pi} \displaystyle\iint_\Sigma \exp\left[-ik(z' \cos\theta + x' \sin\theta)\right] \varphi(M') \; d\Sigma(M) \\ F_1^{-1}\{\tilde{\varphi}(\theta,k)\} = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} d\theta \int_0^\infty \exp\left[ik(z \cos\theta + x \sin\theta)\right] \tilde{\varphi}(\theta,k) \; kdk \end{cases} \tag{5.25}$$

$$= \begin{cases} \varphi(M) & \text{if } M \in \Sigma \\ 0 & \text{if } M \notin \Sigma + C \end{cases}$$

Let us put

$$\left.\begin{aligned} &\gamma(M) = c \sin\delta \text{ if } M \in \Sigma, = 0 \text{ if } M \notin \Sigma + C \\ &g_0(M) = \frac{1}{4\pi} \int_{C^+} F(M,M') \left(\frac{\partial\psi_0}{\partial s}\right) M' \; ds(M') + \frac{1}{4\pi} \int_{C^+} \frac{1}{MM'} \left(\frac{\partial\psi_0}{\partial\nu}\right)_{M'} ds(M') \\ &g_1(M) = \frac{1}{4\pi} \int_{C^+} F(M,M') \left(\frac{\partial\psi_1}{\partial s}\right)_{M'} ds(M') + \frac{1}{4\pi} \int_{C^+} \frac{1}{MM'} \left(\frac{\partial\psi_1}{\partial\nu}\right)_{M'} ds(M') \end{aligned}\right\} \tag{5.26}$$

In the sense of the generalized function theory, we have

$$\frac{1}{MM'} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \int_0^\infty \exp\{ik[(z-z') \cos\theta + (x-x') \sin\theta]\} \; dk \tag{5.27}$$

By substituting this expression into (5.13), we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{\infty} \exp\left[ik(z\cos\theta + x\sin\theta)\right] kdk$$

$$\times \frac{1}{2k}\frac{1}{2\pi}\iint_{\Sigma} \exp\left[-ik(z'\cos\theta + y'\sin\theta)\right] f(M')\, d\Sigma'$$

= right-hand side of (5.13).

Hence:

$$F_1^{-1}\left\{\frac{1}{2k} F_1(f)\right\} = \text{right-hand side of (5.13)},$$

whence

$$f(M) = F_1^{-1}\,\{2kF_1\,\{\text{right-hand side of (5.13)}\}\}$$

$$= F_1^{-1}\,\{2k\tilde{\gamma}(M) + 2k\tilde{g}_0(M) + 2k\tilde{g}_1(M)\}$$

From this we derive:

$$\psi_1(M) = \frac{1}{2\pi}\iint_{\Sigma} \log|MM'|\; f(M')\; d\Sigma(M')$$

$$= \frac{1}{2\pi}\iint_{\Sigma} \log|MM'|\; \{F_1^{-1}\,[2k\tilde{g}_1]\}\; d\Sigma(M') \tag{5.28}$$

$$+ \frac{1}{2\pi}\iint_{\Sigma} \log|MM'|\; \{F_1^{-1}\,[2k\,(\tilde{\gamma} + \tilde{g}_0(M))]\}_{M'}\; d\Sigma(M')$$

Equation (5.28) is a Fredholm equation of the second kind which gives ψ_1 in terms of ψ_0 and of $c \sin \delta$ inside Σ. Let G denote the resolvent kernel of this equation. We obtain:

$$\left.\begin{aligned}
\psi_1(M) &= \iint_\Sigma G(M,M')\,[H_0'(M') + H_0(M')]\,d\Sigma(M') \text{ with} \\
H_0'(M) &= \frac{1}{2\pi}\iint_\Sigma \log|MM'|\,[F^{-1}\{2k\tilde{\gamma}\}]_{M'}\,d\Sigma(M'), \\
H_0(M) &= \frac{1}{2\pi}\iint_\Sigma \log|MM'|\,[F^{-1}\{2k\tilde{g}_0\}]_{M'}\,d\Sigma(M')
\end{aligned}\right\} \tag{5.29}$$

The harmonic function ψ_0 is a linear functional of its values on C. We may write

$$\psi_0(M) = L\{M|\psi_0(C)\} \tag{5.30}$$

It follows from this that

$$H_0(M) = L'\{M|\psi_0(C)\}\ . \tag{5.31}$$

where L' is also a linear functional of $\psi_0(C)$. The generalized Kutta condition on C and the condition which indicates that ψ = constant on E_1EE_0 may be written in the form:

$$a(s)\,\frac{\partial\psi}{\partial s} + b(s)\,\frac{\partial\psi}{\partial\nu} = 0 \ \underline{\text{on C}} \tag{5.32}$$

By taking into account Equations (5.29)-(5.32), we obtain:

$$\left.\begin{aligned}
& a(s)\,\frac{\partial}{\partial s}\,L\{M|\psi_0(C)\} + b(s)\,\frac{\partial}{\partial\nu}\,L\{M|\psi_0(C)\} \\
& + a(s)\,\frac{\partial}{\partial s}\iint_\Sigma G(M,M')\,L'\{M'|\psi_0(C)\}\,d\Sigma(M') \\
& + b(s)\,\frac{\partial}{\partial\nu}\iint_\Sigma G(M,M')\,L'\{M'|\psi_0(C)\}\,d\Sigma \\
& = -\left[a(s)\,\frac{\partial}{\partial s} + b(s)\,\frac{\partial}{\partial\nu}\right]\iint_\Sigma G(M,M')\,H_0'(M')\,d\Sigma'(M') \ \underline{\text{on C}}
\end{aligned}\right\} \tag{5.33}$$

This is an integral equation of the second kind which determines $\psi_0(C)$. When ψ_0 is determined on C, ψ_1 is determined by Equation (5.28) and

$$\psi = \psi_0 + \psi_1 \tag{5.34}$$

The preceding proves the existence of the solution of the problem.* But the writer dares not claim that the above method should be used for computing the solution.

The problem of the infinitely thin double model does not appear to be simpler than that of the double model of a real ship. Despite its shortcomings, the Casal theory is probably the best one to use when the purpose is simply to give a qualitative account of the phenomena involved in ship maneuvering.

A TENTATIVE THEORY FOR NON-INFINITELY THIN SURFACE SHIPS

We shall consider only the case of a ship in uniform motion of translation and assume that the relative motion is steady. Furthermore, we assume that the boundary condition on the free surface may be linearized.

Let Σ_0 denote the wetted surface of the ship. Since the boundary condition on the free surface is linearized, Σ_0 is the part of the hull located below the plane z = 0 of the free surface at rest.

The ship moves with speed c ($=V_E$) in the positive x_1-direction. For the sake of simplicity, the hull heel is neglected. The longitudinal plane of symmetry of Σ_0 is thus vertical. Let (Oz, Ox) be this plane. The drift

---

*The "proof" given here is incomplete, however, since it has not been established that the determinant of Equation (5.28) is other than zero.

$\delta = (Ox_1, Ox)$ is positive and small enough for no separation to occur.

Vector Potentials

To apply the method used in Chapter 4 (Section entitled "double model in an oblique, uniform motion of translation"), we shall consider below a double model Σ which lower half is the real hull Σ_0, the upper half being the image Σ_1 of Σ_0 with respect to the plane $z = 0$ of the free surface at rest. Let Σf_0 be the real free vortex sheet and Σf_1 its image. In what follows, Σ_f denotes the union $\Sigma f_0 \cup \Sigma f_1$.

Any free vortex filament Lf_0 located on Σf_0 comes from infinity downstream, and ends at a point B_o located on the arc E_0S_0S of the contour of the centerplane of the ship. Its support coincides with a streamline of the relative motion, but it is in a direction opposite to the relative velocity $\vec{V}_R$. This free vortex filament Lf_0 is a part of a U-shaped one $L'' = L_{f_0}+L''_b+L_{f_1}$. The arc L''_b is bound on Σ^+, the portside of Σ; beginning at B_0, it intersects the arc C^+ of the waterline C and ends at B_1, the image of B_0 on the arc SS_1E_1. The third part Lf_1 is the image of Lf_0 on Σf_1, image of Σf_0. The intensity of L" is a constant depending on the abscissa of B_0 on the contour of the centerplane.

There exists on Σ another family of vortex filaments L'. Each filament L' is, in fact, a vortex ring. On Σ^-, the starboardside of Σ, it is in the direction from $B_1\varepsilon SS_1E_1E$ towards $B_0\varepsilon SS_0E_0E$, B_1 being still the image of B_0. On Σ^+, L' is in the direction from B_0 towards B_1. The intensity of any vortex ring is a constant depending only on the position of B_0.

We have thus two bound vortex sheets on Σ. One consists of the union of the vortex rings; it extends on the whole surface Σ. The second one is located on Σ^+, behind a certain arc E_0E_1. Let ε be the uniform, infinitely small thickness of the vortex sheets. The vorticity at a point P_0 on Σ_0 or Σf_0 is

$$\vec{\omega}(P_0) = \frac{1}{\varepsilon} \vec{T}_0, \text{ with } \vec{T}_0 = (\xi_0, \eta_0, \zeta_0) \tag{5.35}$$

At P_1, image of P_0, we have

$$\vec{\omega}(P_1) = \frac{1}{\varepsilon} \vec{T}_1, \text{ with } \vec{T}_1 = (-\xi_0, -\eta_0, \zeta_0) \tag{5.35'}$$

The two vectors

$$\vec{J}'(M) = \frac{1}{4\pi} \iint_{\Sigma} \frac{\vec{T}'_b(P)}{PM} \, d\Sigma \, (P) \tag{5.36}$$

$$\vec{J}''(M) = \frac{1}{4\pi} \iint_{\Sigma_{f_0}} \frac{\vec{T}_{f_0}(P)}{PM} \, d\Sigma_{f_0} \, (T) + \frac{1}{4\pi} \iint_{\Sigma^+} \frac{\vec{T}''_b(P)}{PM} \, d\Sigma^+ \, (P)$$

$$+ \frac{1}{4\pi} \iint_{\Sigma_{f_1}} \frac{\vec{T}_{f_1}(P)}{PM} \, d\Sigma_{f_1} \, (P) \tag{5.37}$$

are both divergenceless. They play the part of vector potentials and generate the velocities

$$\vec{V}'(M) = \nabla \wedge \vec{J}'(M), \quad \vec{V}''(M) = \nabla \wedge \vec{J}''(M) \tag{5.38}$$

The total velocity induced by the vortex distribution is

$$\vec{V}_2 = \vec{V}' + \vec{V}'' \tag{5.39}$$

It is defined in the whole space, above the plane $z = 0$ as well as below this plane and inside Σ as well as outside Σ. But it is only inside $D_e \cap (z < 0)$ that $\vec{V}_2$ has a physical meaning, D_e being the domain exterior to Σ.

Each of the two vortex sheets defined by (5.36) and (5.37), respectively, is equivalent to a normal dipole distribution. We may write:

$$\vec{V}'(M) = \nabla \Phi'(M), \quad \Phi'(M) = \frac{-1}{4\pi} \iint_{\Sigma} \mu'_b(P) \frac{\partial}{\partial n_P} \frac{1}{PM} \, d\Sigma(P) \tag{5.40}$$

$$\vec{V}''(M) = \nabla \Phi''(M), \quad \Phi''(M) = \frac{-1}{4\pi} \iint_{\Sigma} \mu''_b(P) \frac{\partial}{\partial n_P} \frac{1}{PM} \, d\Sigma^+ \, (P)$$

$$- \frac{1}{4\pi} \iint_{\Sigma_f} \mu_f(P) \frac{\partial}{\partial n_P} \frac{1}{PM} \, d\Sigma_f(P) \tag{5.41}$$

But we may also write:

$$\vec{V}_2 = \vec{V}'_2 + \vec{V}_f, \text{ where} \tag{5.42}$$

$$\vec{V}_f = \nabla \Phi_f, \quad \Phi_f(M) = - \frac{1}{4\pi} \iint_{\Sigma_f} \mu_f(P) \frac{\partial}{\partial n_P} \frac{1}{PM} \, d\Sigma_f(P) \tag{5.43}$$

$$\vec{V}'_2 = \nabla\Phi'_2, \quad \Phi'_2(M) = -\frac{1}{4\pi}\iint_\Sigma \mu(P)\frac{\partial}{\partial n_P}\frac{1}{PM}\,d\Sigma\ (P) \tag{5.44}$$

$$\mu(P) = \begin{cases} \mu^+ (P) = \mu' (P) + \mu''_b (P) \text{ on } \Sigma^+ \text{ (port side)} \\ \\ \mu^- (P) = \mu' (P) \text{ on } \Sigma^- \text{ (starboard side)} \end{cases} \tag{5.45}$$

One has

$$\mu''_b = 0 \text{ ahead of a certain arc } E_0E_1 \text{ on } \Sigma^+.$$

As in Chapter 4, let λ be the curvilinear abscissa along the contour of the centerplane of Σ, with $\lambda = 0$ at E, $\lambda < 0$ on SS_1E_1E, $\lambda > 0$ on EE_0S_0S. We have

$$\frac{d\Gamma_f}{d\lambda} = \frac{d\mu_f}{d\lambda} = \frac{d\Gamma''}{d\lambda} = \frac{d}{d\lambda}(\mu^+ - \mu^-) \text{ at } B_0 \text{ on } E_0S_0S \tag{5.46}$$

If the surface Σ_f were known and also, on Σ_f, the lines μ_f = constant (that is the vortex filaments L_f) and the function $\frac{d\lambda_f}{d\lambda}$ (λB_0), then Φ_f could be calculated on the inner side Σ_i of Σ. By means of equation

$$(\Phi'_2 + \Phi_f)_{M_i} = c\, x_1(M_i) + \text{constant} \tag{5.47}$$

one could determine μ on Σ. The potential Φ'_2 so defined is the solution of a Dirichlet interior problem. The Fredholm equation giving μ is regular and has a unique solution. The velocity $\vec{V}_2$ (equation (5.42)), satisfies the boundary condition at the hull, namely

$$\vec{V}_2 \cdot \vec{n} = \vec{n} \cdot \vec{c} \text{ on } \Sigma_e \tag{5.48}$$

and according to (5.47), $\vec{V}_2 \cdot \vec{n} = \vec{n} \cdot \vec{c}$ on Σ_i since $\frac{\partial}{\partial n}\Phi'_2$, $\frac{\partial}{\partial n}\Phi_f$ are continuous across Σ.

In fact, the surface Σ_f and the lines L_f are not known. But, having regard to the smallness of δ, one may assume that they are half-straight lines parallel to $\vec{c}$ (in the opposite direction on Σf_0 and in the direction of $\vec{c}$ on Σf_1). One may also assume that they start from the contour of the centerplane (except from E_0EE_1). For larger values of δ, the line $\mathcal{B}$ would be displaced towards portside on the stern region and separation could occur, particularly in the case of a flat bottom.

The condition determining $\frac{d\Gamma_f}{d\lambda}$ is the generalized Kutta condition. It expresses in the present case, as mentioned in Chapter 4, the equality

$$|V_R^+ (B_0)| = |V_R^- (B_0)| \text{ on } E_0S_0S \tag{5.49}$$

The exact treatment of this condition is difficult and the approximate solution given in Chapter 4 is rather crude.

Wave field

Since we are now dealing with a surface ship, the real velocity in the liquid domain cannot be reduced to just $\vec{V}_2$ in equation (5.42). The total velocity $\vec{V}$ must satisfy the free-surface condition (here we assume that it can be linearized) and have on the hull Σ_0 and the new surface of the vortex sheet Σf_0 properties analogous to those of $\vec{V}_2$.

In fact, the free vortex filaments coincide with arcs of relative motion streamlines. Therefore, they exert on the flow a zero system of forces and contribute nothing to the waves. (This would hold even if the free-surface condition were not linearized). The vortices pearce the free surface along its intersection with the hull. To close them, the free-surface condition being linearized, one may still consider the images Σ_1 of Σ_0 and Σf_1 of Σf_0, and the Φ_2' expression in terms of μ and also that of Φ_f in terms of μ_f are both unchanged. But we have to add to Φ_2' (or to $\Phi_2 = \Phi_2' + \Phi_f$) a velocity potential Φ_1 such that the free-surface condition be fulfilled. Putting $k_o = g/c^2$, where g = the acceleration of gravity, and introducing a vanishing friction force $- \varepsilon' \vec{V}$ per unit mass of liquid, we must have:

$$(k_o \frac{\partial}{\partial z} - \frac{\varepsilon'}{c} \frac{\partial}{\partial x_1} + \frac{\partial^2}{\partial x_1^2}) (\Phi_2 + \Phi_1)|_{z=0} \equiv 0. \tag{5.50}$$

We have already remarked that the z-component of the velocity induced by the vorticity is null at z = 0 (because the free-surface condition is linearized). We note, furthermore, that $\frac{\partial}{\partial x1} \Phi_f = 0$ for z = 0 if the L_f's are parallel to $\vec{c}$. But for reasons already mentioned, Φ_f contributes zero to the x_1-component of $\vec{V}$. Finally (5.50) is reduced to

$$\left(k_o \frac{\partial}{\partial z} - \frac{\varepsilon'}{c}\frac{\partial}{\partial x_1} + \frac{\partial^2}{\partial x_1^2}\right)\Phi_1\Big|_{z=0} \equiv - \frac{\partial^2}{\partial x_1^2}\Phi_2'\Big|_{z=0} = 0. \tag{5.51}$$

Let F_1, F_1^{-1} be the operators of the Fourier transform of a function of (x, y) or (x_1, y_1). Let us put

$$\left.\begin{matrix} I_1(M) \\ \\ I_1^*(M) \end{matrix}\right\} = \exp\left(\pm ik\,(x_1\cos\theta + y_1\sin\theta)\right) \tag{5.52}$$

and, P_0 being a point on Σ_0, P_1 its image, we denote their coordinates

$$x_1', \; y_1', \; \pm z'$$

and, putting for the sake of brevity

$$I_1 = I_1(M), \; I_1'^* = I_1^*(P_0) = I_1'^*(P_1)$$

we have

$$I_1 I_1'^* = \exp(ik\bar{\omega}), \text{ where } \bar{\omega} = (x_1 - x_1')\cos\theta + (y_1 - y_1')\sin\theta \tag{5.53}$$

Let, along the waterline WL, in the plane z = -0,

$\vec{n}$ be the unit vector normal to Σ in the inward direction, (5.54)

$\vec{s}$ the unit vector tangent to WL in the positive direction with respect to the z-axis,

$\vec{\tau}$ one of the two unit vectors tangent to Σ_0 and normal to WL.

We put

$$\alpha = \cos(0x_1, \vec{n}), \; \beta = \cos(0x_1, \vec{s}), \; \gamma = \cos(0x_1, \vec{\tau}) \tag{5.54'}$$

For the sake of brevity we shall often write $\hat{f}(\theta, k)$ for $F_1\{f(x_1, y_1, 0)\}$. Equation (5.51) gives

$$\hat{\Phi}_1(\theta, k) = \frac{\sec^2\theta}{k(k - k_o\sec^2\theta + i\,(0+))} F_1\left\{\frac{\partial^2}{\partial x_1^2}\Phi_2'\Big|_{z=0}\right\}. \tag{5.55}$$

Let FS_i, resp. FS_e, be the part of the plane z = -0 interior, resp. exterior, to the intersection WL of this plane with the hull Σ_0, and $\Phi_2'^{i}$, resp. $\Phi_2'^{e}$, the determination of Φ_2' within FS_i, resp. FS_e. We have

$$\left.\begin{aligned} &(\Phi_2'^{i} - \Phi_2'^{e})_{M'} = -\mu(M') \\ &\left(\frac{\partial}{\partial x_1}\Phi_2'^{i} - \frac{\partial}{\partial x_1}\Phi_2'^{e}\right)_{M'} = \left(\beta\frac{\partial\mu}{\partial s} + \gamma\frac{\partial\mu}{\partial\tau}\right)_{M'} \end{aligned}\right\} \quad (\text{any } M' \in WL) \qquad (5.56)$$

Furthermore:

$$F_1\left\{\frac{1}{P_0M}\right\} = F_1\left\{\frac{1}{P_1M}\right\} = \exp(kz')I_1(M')/k \text{ for } M \in FS_i + FS_e \qquad (5.57)$$

$$\frac{\partial}{\partial x_{P_0}}\frac{1}{P_0M} = \frac{\partial}{\partial x_{P_1}}\frac{1}{P_1M} \text{ for } M \in FS_i + FS_e \qquad (5.57')$$

the latter equation being due to

$$\frac{\partial}{\partial z_{P_0}} = -\frac{\partial}{\partial z_{P_1}}, \ \cos(0z_1\ \vec{n}_{P_0}) = -(\cos 0z,\ \vec{n}_{P_1}).$$

On the other hand, because of the discontinuities, equations (5.56), of Φ_2' and $\frac{\partial\Phi_2'}{\partial x_1}$ across Σ, we have

$$F_1\left\{\frac{\partial^2}{\partial x_1^2}\Phi_2'\Big|_{z=0}\right\} = -k^2\cos^2\theta\,\Phi_2' + \frac{1}{2\pi}\int_{WL} I_1'^{*}\left(\left(\beta\frac{\partial\mu}{\partial s} + \gamma\frac{\partial\mu}{\partial\tau}\right)_{M'} + ik\cos\theta\,\mu(M')\right)dy_1(M') \qquad (5.58)$$

From (5.57'), we derive

$$\hat{\Phi}_2'(\theta, k) = F_1\left\{\iint_{\Sigma_0}\mu(M')\left(\frac{\partial}{\partial n_{P_0}}\frac{1}{P_0M} + \frac{\partial}{\partial n_{P_1}}\frac{1}{P_1M}\right)_{z=0} d\Sigma_0(P_0)\right.$$

$$= (\text{after } 5.57) \iint_{\Sigma_0} -\frac{\mu}{4\pi}(P_0')\frac{\partial}{\partial n_{P_0}}(2\exp(kz')\,I_1'^{*}/k)\,d\Sigma_0(P_0) \qquad (5.59)$$

Let Φ_1' be the contribution to Φ_1 from the first term on the right-hand side of (5.58) and Φ_1'' that from the second one. According to (5.55), we have

$$\hat{\Phi}_1'(\theta,k)=\iint_{\Sigma_0} \frac{\mu(P_0)}{4\pi}\frac{\partial}{\partial n_{P_0}} (2\ \exp(kz')\ I_1'^*)\ \frac{1}{k - k_o \sec^2\theta + i(0+)} d\Sigma_0(P_0) \qquad (5.60)$$

$$\hat{\Phi}_1''(\theta,k)=\frac{1}{2\pi}\int_{WL} -(\beta\frac{\partial\mu}{\partial s}+\gamma\frac{\partial\mu}{\partial\tau} + ik\ \cos\theta\mu)_{M'} \frac{\sec^2\theta}{k-k_o\sec^2\theta+i(0+)}\frac{\partial}{\partial n_{M'}} I_1'^* dy_1(M') \qquad (5.61)$$

Now, let ϕ denote the velocity potential due to a unit Kelvin source located at $P_0(x_1', y_1', z')$, $z' < 0$, and moving with the same speed c as the ship in the positive x_1-direction. We may write

$$\phi(M, P_0) = \frac{1}{4\pi} (\frac{-1}{P_0M} + \frac{-1}{P_1M} + \psi_1' (M, P_1)) \qquad (5.62)$$

$$= \frac{1}{4\pi} (- \frac{1}{P_0M} + \frac{1}{P_1M} + \psi_1'' (M, P_1)) \qquad (5.62')$$

We have

$$\hat{\psi}_1' = \frac{2I_1'^* \exp(kz')}{k - k_o\sec^2 \theta + i(0+)} \qquad (5.63)$$

$$\hat{\psi}_1'' = \frac{2\ k_o\ \sec^2 \theta I_1'^* \exp(kz')}{k(k - k_o\ \sec^2\theta + i(0+))} \qquad (5.63')$$

Be combining (5.60) and (5.63) we obtain

$$\hat{\Phi}_1' = \iint_{\Sigma 0} \frac{1}{4\pi} \mu(P_0) \frac{\partial}{\partial x_{P_0}} \hat{\psi}_1'\ d\Sigma_0(P_0)$$

whence

$$\Phi_2' + \Phi_1' = \iint_{\Sigma 0} \mu(P_0) \frac{\partial}{\partial n_{P_0}} \phi\ (M, P_0)\ d\Sigma(P_0) \qquad (5.64)$$

On the other hand, by combining (5.61) and (5.63'), we derive:

$$\Phi_1'' = \frac{1}{k_0} \int_{WL} (-\beta\frac{\partial\mu}{\partial s} + \frac{\partial\mu}{\partial\tau})_{M'} + \mu(M') \frac{\partial}{\partial x_1 M'}) \ \phi\ (M, M')\ dy_1(M') \qquad (5.65)$$

The total velocity potential (for $z < 0$) is therefore

$$\Phi(M) = \Phi_f + \Phi_2' + \Phi_1 = \Phi_f + (\Phi_2' + \Phi_1') + \Phi_1''$$

$$= \frac{-1}{4\pi} \iint_{\Sigma_f} \mu_f(P) \frac{\partial}{\partial n_P} \frac{1}{PM} d\Sigma_f$$

$$+ \iint_{\Sigma_0} \mu(P_0) \frac{\partial}{\partial n_{P_0}} \phi(M, P_0) \, d\Sigma_0(P_0) \tag{5.66}$$

$$+ \frac{1}{k_0} \int_{WL} \left(-\beta \frac{\partial\mu}{\partial s} + \frac{\partial\mu}{\partial\tau}\right)_{M'} + \mu(M') \frac{\partial}{\partial x_{1M'}}\Big) \, \phi\,(M, M') \, dy_1(M')$$

We recall that μ_f and μ are connected to each other by the generalized Kutta condition and that μ is determined by the boundary condition at the hull, namely

$$(\Phi_f + \Phi_2' + \Phi_1)_{M_i} \equiv c\, x_1(M_i) \text{ on } \Sigma_i \tag{5.67}$$

We also recall that $\Sigma_f = \Sigma_{f_0} + \Sigma_{f_1}$. The second term on the right-hand side represents a Kelvin normal dipole distribution on the open surface Σ_0 of the hull. The line integral represents the sum of a limiting Kelvin source distribution and of a limiting Kelvin longitudinal dipole distribution along the waterline. These last two distributions are dissymetric.

We remind the reader of the assumptions:

- The intersection of Σ_{f_0} with the hull Σ_0 is supposed to be in the centerplane;
- The drift angle is supposed to be small enough for the flow to be non separated;
- The heel is neglected;
- The free-surface condition is linearized.

There is no need to say that in spite of very important simplifications, solving equation (5.67) would be considerably intricate.

CHAPTER 6: SMALL MOTIONS OF A BODY ABOUT A UNIFORM MOTION OF TRANSLATION

Let us first define Small Motions as studied in the present chapter.

The fluid is unbounded and almost inviscid. The speed $\vec{V}_E(0)$ of the origin 0 of the system S of axes $0(x_1, x_2, x_3)$ moving with the body is, in the mean, constant and parallel to the x_1'-axis of the fixed system S' of axes $0'$ (x_1', x_2', x_3'). The components (u_1, u_2, u_3) of $\vec{V}_E(0)$ on the moving axes are such that, with c a positive constant, the quantities $(u_1-c)^2$, u_2^2, and u_3^2 are negligibly small with respect to c^2. The angular velocity $\vec{\Omega}_E$ of the system S with respect to S' is zero in the mean. Let L denote the length of the body in the x_1-direction and let u_4, u_5, u_6 be the components of $\vec{\Omega}_E$ on the moving axes. The quantities $u_4 \frac{L}{2}$, $u_5 \frac{L}{2}$, $u_6 \frac{L}{2}$ are small enough for their squares to be negligibly small with respect to c^2. Free vortices are shed, but the "shedding line" $\mathcal{B}$ is supposed to be in a fixed position on the hull Σ of the body.

The above assumptions define a small motion of the body about a uniform motion of translation of speed c in the positive x_1'-direction.

Nevertheless there are two kinds of such *small motions*.

In motions of the first kind, the line $\mathcal{B}$ is approximately in a plane parallel to the (x_1, x_2)-plane and $\vec{\Omega}_E$ is approximately parallel to the x_2-axis. The support Σ_f of the free vortices can be considered as a surface generated by the curves close to half-straight lines in the negative x_1'-direction which will be considered horizontal. The small motions of the first kind are therefore analogous to small motions of a submarine in the vertical plane. The rotational wake is confined on the surface Σ_f.

In motions of the second kind, the free vortices are shed from a line $\mathcal{B}$ analogous to the arc $E_0S_0SS_1E_1$ of the contour C intersected in the hull of a double model by its longitudinal plane of symmetry. Nevertheless, the body is not assumed to be symmetrical with respect to some horizontal plane. It will be seen that the motions of the second kind differ from those of the first kind in that the vortex wake generated by the body is three-dimensional and thus not confined on a certain surface.

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SMALL MOTIONS OF THE FIRST KIND

Kinematics

Let B_0 (to port) and B'_0 (to starboard) be the two ends of the shedding line $\mathcal{B}$. Let σ_M be a curvilinear abscissa on $\mathcal{B}$ and $-\ell$, $+\ell$ the values of σ at B_0 and B'_0 respectively. Let t_0 be the time when the motion begins, let $\mathcal{B}_f$ be a line on Σ_f infinitely close to $\mathcal{B}$ (Figure 12), and let $P(t_0, \sigma)$ be the fluid point which was at t_0 at the point $B_f(\sigma)$.

At t this fluid point is at a position $B^t_{t_0}(\sigma)$ such that

$$\overrightarrow{B_f(\sigma)\ B^t_{t_0}(\sigma)} = \int_{t_0}^{t} \vec{V}_R\ (P(t_0, \sigma), t')\ dt' \tag{6.1}$$

When - with t fixed - σ varies from $-\ell$ to $+\ell$, the line described by the point $B^t_{t_0}(\sigma)$ is the boundary of $\Sigma_f(t)$ in the negative x'-direction. Let $\mathcal{B}^t_{t_0}$ denote this line. Let P (τ, σ) be the fluid point which was at time τ at $B_f(\sigma)$. At t, this fluid point is at a position $B^t_\tau(\sigma)$ such that

$$\overrightarrow{B_f(\sigma)\ B^t_\tau(\sigma)} = \int_{\tau}^{t} \vec{V}_R\ (P(\tau, \sigma), t')\ dt' \tag{6.2}$$

When-with τ and t fixed-σ varies from $-\ell$ to $+\ell$, the line described by $B^t_\tau(\sigma)$ is a line $\mathcal{B}^t_\tau$ located on $\Sigma_f(t)$ between $\mathcal{B}_f$ and $\mathcal{B}^t_{t_0}$; $\mathcal{B}_f$ is the line $\mathcal{B}_{t-0}$. The edges of $\Sigma_f(t)$ are the arcs described by the points $B^t_\tau(-\ell)$ and $B^t_\tau(+\ell)$.

The above definition of the free vortex sheet $\Sigma_f(t)$ is general and involves no assumption concerning the nature of the motion of the body except that for the fixity of $\mathcal{B}$ on Σ.

Let us consider now the parts Σ_1 and Σ_2 of Σ which are the supports at t of the bound vortex filaments L_{b_1}, L_{b_2} beginning or ending on $\mathcal{B}$. The part Σ_2 is included in the suction side and bounded by $\mathcal{B}$ and by the bound vortex filament beginning at B_0 and ending at B'_0. In contrast, Σ_1 is partly on the suction side and partly on the pressure side. Its frontier consists of $\mathcal{B}$ on the pressure side, of the bound vortex $B_0\ B'_0$ on the suction side,

and of a certain vortex filament (coming from a certain point of $\mathcal{B}$ and ending at the same point) on both sides. The Σ_3 part is covered by closed, bound vortices only. The general disposition of the bound vortices is similar to that represented in Figure 5 for wings of finite thickness in a steady motion of translation. Any vortex filament L_1 whose bound part is located on Σ_1 is continued on Σ_f by a free vortex L_{f_1}. The free arc L_{f_1} of L_1 ends at the beginning B of L_{b_1} and begins at the end B' of L_{b_1}. The point B is also the end of L_{f_2} and the beginning of L_{b_2} while B' is also the beginning of L_{f_2} and the end of L_{b_2}. On Σ_f, the supports of L_{f_1} and L_{f_2} coincide and the intensity of the union L_f of L_{f_1} and L_{f_2} is the sum of the intensities of L_{b_1} and L_{b_2}. It is suitable for the following to consider that L_{b_1} and L_{b_2} are closed by a vortex filament located on the arc B'B of $\mathcal{B}$ and that there exists on $B_f'B_f$ a filament of intensity $d\sigma\Gamma_1(\sigma, t) + d\sigma\Gamma_2(\sigma, t)$, σ being the abscissa of B on $\mathcal{B}$. As $B_f'B_f$ is infinitely close to B'B and in the direction opposite to BB', the effect of the fictitious vortex filament B'B is cancelled by that of the vortex filament B_fB_f'. The advantage of this procedure is that the vortex filament of intensity $-d\sigma[\Gamma_1(\sigma, t) + \Gamma_2(\sigma, t)]$ located on the arc $B_f(\sigma)B_f(\sigma + d\sigma)$ is in a position independent of t, whereas the positions of L_{b_1} on Σ_1 and of L_{b_2} on Σ_2 may vary with t.

During the time interval $(\tau, \tau + d\tau)$, the intensity of the vortex filament located on the arc $B_f(\sigma)B_f(\sigma + d\sigma)$ has an increment $-d_\tau d_\sigma[\Gamma_1(\sigma, t) + \Gamma_2(\sigma, t)]$ and a free vortex filament intensity $+d_\tau d_\sigma[\Gamma_1(\sigma, t) + \Gamma_2(\sigma, t)$ appears at $\tau + d\tau$ on the closed contour $B_f(\sigma)\ B_\tau^{\tau+d\tau}(\sigma)\ B_\tau^{\tau+d\tau}(\sigma+d\sigma)$ $B_f(\sigma + d\sigma)\ B_f(\sigma)$. This free vortex filament moves with the fluid and at t it becomes

$$B_\tau^t(\sigma)B_{\tau + d\tau}^t(\sigma)B_{\tau + d\tau}^t(\sigma + d\sigma)B_\tau^t(\sigma + d\sigma)B_\tau^t(\sigma)$$

Its intensity is unchanged and thus equal to $d_\tau d_\sigma[\Gamma_1(\sigma, \tau) + \Gamma_2(\sigma, \tau)]$. The points $B_{\tau + d\tau}^t(\sigma)$ and $B_\tau^t(\sigma)$ are, of course, on the line $B_f(\sigma)B_{t_0}^t(\sigma)$. This allows us to describe the structure of the free vortex sheet $\Sigma_f(t)$ at time t as follows:

Let $\Sigma_{f_1}(t)$, $\Sigma_{f_2}(t)$ be the two sides of $\Sigma_f(t)$; $\Sigma_{f_1}(t)$ and $\Sigma_{f_2}(t)$ are the continuations at t of Σ_1 and Σ_2, respectively. Let $\vec{n}_f$ denote the unit vector normal to Σ_f in the direction from $\Sigma_{f_2}(t)$ toward $\Sigma_{f_1}(t)$. Let M_f be the point of $\Sigma_f(t)$ which coincides with $B^t_\tau(\sigma)$ and M_{f_1}, M_{f_2} the points defined by $\overrightarrow{M_fM_{f_1}} = -\vec{n}_{M_f}(0+)$, $\overrightarrow{M_fM_{f_2}} = \vec{n}_{M_f}(0+)$. Similarly, let M'_f be the point which coincides with $B^t_\tau(\sigma')$. When σ' increases from $-\ell$ to σ, M'_f generates an arc $C(\sigma; \tau, t)$. Let $C_1(\sigma; \tau, t)$ and $C_2(\sigma; \tau, t)$ denote the arcs described by M'_{f_1} and M'_{f_2}, respectively. Let $S(\sigma; \tau, t)$ be the open surface generated by $M'_{f_2}M'_{f_1}$. It is normal to $\Sigma_f(t)$ and its edge is the closed contour

$$M_{f_1}(\sigma)\ B^t_{1_\tau}(-\ell)\ B^t_{2_\tau}(-\ell)\ M_{f_2}(\sigma)\ M_{f_1}(\sigma) \tag{6.3}$$

The surface $S(\sigma; \tau, t)$ is intersected by all the free vortex filaments which encounter $C(\sigma; t, t)$. That of these vortex filaments which passes through M'_f has at this point the intensity

$$\int_{t_0}^{\tau} d_{\tau'} d_{\sigma'}[\Gamma_1(\sigma',\tau') + \Gamma_2(\sigma',\tau')] = d_{\sigma'}[\Gamma_1(\sigma', \tau) + \Gamma_2(\sigma', \tau)]$$

The circulation of $\vec{V}$ in the circuit (6.3) is therefore

$$\int_{-\ell}^{\sigma} d_{\sigma'}[\Gamma_1(\sigma', \tau) + \Gamma_2(\sigma', \tau)] = \Gamma_1(\sigma, \tau) + \Gamma_2(\sigma, \tau) \tag{6.4}$$

The velocity $\vec{V}$ generated by all the bound and free vortex filaments is irrotational in the domain $[D_e - \Sigma_f(t)]$, D_e being the exterior of Σ at t:

$$\vec{V}(M, t) = \nabla\Phi(M, t) \quad \text{inside} \quad D_e - \Sigma_f(t) \tag{6.5}$$

To any point $M_f(\sigma)$ on $\Sigma_f(t)$, there corresponds one, and only one, point $B(\sigma)$ defined by

$$\overrightarrow{B(\sigma)\ M_f(\sigma)} = \int_{\tau}^{t} \vec{V}_R(P(\sigma, \tau), t')\ dt' \tag{6.6}$$

Here P is the fluid point located at $M_f(\sigma)$ at t and τ is the time when this fluid point was on $\mathcal{B}$ at a certain point $B_f(\sigma)$ <u>inside</u> the free vortex sheet. We have

$$\Phi(M_{f_2}(\sigma), t) - \Phi(M_{f_1}(\sigma), t) = \Gamma_1(\sigma, \tau) + \Gamma_2(\sigma, \tau) \tag{6.7}$$

The vortex distribution $\mathcal{D}$ which generates Φ may be written in the form:

$$\left.\begin{aligned}
\mathcal{D} &= \mathcal{D}_1 + \mathcal{D}_2' + \mathcal{D}_2'', \text{ with} \\
\mathcal{D}_1 &= \left[\left(\Sigma, \frac{\vec{T}_1(t)}{\varepsilon}\right) + (D_i, 2\vec{\Omega}_E(t))\right] \\
\mathcal{D}_2' &= \left(\Sigma, \frac{\vec{T}_2'(t)}{\varepsilon}\right) \\
\mathcal{D}_2'' &= \left[\left(\Sigma, \frac{\vec{T}_2''(t)}{\varepsilon}\right) + \left(\Sigma_f, \frac{\vec{T}_f(t)}{\varepsilon}\right)\right]
\end{aligned}\right\} \tag{6.8}$$

$(\mathcal{D}_1 + \mathcal{D}_2')$ is identical with the vortex distribution which is kinematically equivalent to the given body when no free vortices are shed. Let $\vec{V}_1$ and $\vec{V}_2'$ be the velocities induced by $\mathcal{D}_1$ and $\mathcal{D}_2'$, respectively. We have

$$\left.\begin{aligned}
&\vec{V}_1 \equiv 0 \text{ inside } D_e, \text{ curl } \vec{V}_1 = 2\vec{\Omega}_E(t) \text{ within } D_i \\
&(\vec{V}_E - \vec{V}_1)_{M,\ t} = \nabla\Phi'(M,\ t) \text{ within } D_i \\
&\frac{\partial\Phi}{\partial n} \equiv (\vec{n} \cdot \vec{V}_E)_{M,\ t} \text{ on } \Sigma_e \\
&\vec{V}_2' = \nabla\Phi_2', \text{ with } \nabla\Phi_2'(M) = (\vec{V}_E - \vec{V}_1) \text{ within } D_i \\
&\Phi_2'(M_i,\ t) = \Phi'(M_i,\ t) \text{ on } \Sigma_i \text{ at } t \\
&\Phi' = \sum_{j=1}^{6} u_j(t)\ \Phi_j'(M) \text{ everywhere within } D_i
\end{aligned}\right\} \qquad (6.9)$$

then

$$\mathcal{D}_2' = (\Sigma,\ \mu'\vec{n}) \qquad (6.10)$$

The velocity $\vec{V}_2'$ induced by $\mathcal{D}_2'$ is within D_i

$$\vec{V}_2' \equiv \vec{V}_E - \vec{V}_1 \qquad (6.11)$$

The total velocity within D_i is thus:

$$\vec{V} = \vec{V}_1 + (\vec{V}_E - \vec{V}_1) \equiv \vec{V}_E \text{ within } D_i \qquad (6.11a)$$

and, since its normal component is continuous through Σ, the boundary condition on Σ_e

$$(\vec{V} \cdot \vec{n})_{M_e,\ t} \equiv \vec{V}_E(M,\ t) \qquad (6.12)$$

is automatically satisfied.

The distribution $\mathcal{D}''_2$ is equivalent to a normal dipole distribution on $\Sigma + \Sigma_f(t)$. This normal dipole distribution generates the velocity potential

$$\left.\begin{aligned}
&\Phi''_2 + \Phi_f \text{ with} \\
&\Phi''_2(M, t) = \frac{-1}{4\pi} \iint_{\Sigma} \mu''(M', t) \frac{\partial}{\partial n_{M'}} \frac{1}{MM'} \, d\Sigma(M') \\
&\Phi_f(M, t) = \frac{-1}{4\pi} \iint_{\Sigma_f} \mu_f(M'_f(\sigma), t) \frac{\partial}{\partial n_{M'}} \frac{1}{MM'_f} \, d\Sigma_f(M'_f(\sigma))
\end{aligned}\right\} \tag{6.13}$$

The discontinuity of

$$\Phi = \Phi'_2 + \Phi''_2 + \Phi_f \tag{6.14}$$

through Σ_f, from $M_{f_1}(\sigma)$ to $M_{f_2}(\sigma)$ is equal to $\mu_f(M'_f(\sigma), t)$. According to Equation (6.8), we have

$$\mu_f(M'_f(\sigma, t)) = (\Gamma_1 + \Gamma_2)_{\sigma,\, \tau} = \Gamma_f(B_f(\sigma), \tau) \tag{6.15}$$

Furthermore, because of (6.11) and (6.11a), we must have

$$\Phi''_2(M, t) = - \Phi_f(M, t) \text{ within } D_i \tag{6.16}$$

Hence μ'' is the solution of the Fredholm equation of the second kind

$$- \frac{1}{2} \mu''(M, t) - \frac{1}{4\pi} \iint_{\Sigma} \mu''(M', t) \frac{\partial}{\partial n_{M'}} \frac{1}{MM'} \, d\Sigma(M) = - \Phi_f(M, t) \quad \text{on } \Sigma \tag{6.17}$$

The vortex $\frac{\vec{T}}{\varepsilon}$ on Σ is finally given by

$$\vec{T}(M, t) = -(\vec{n} \wedge \vec{V}_R)_{M_e, t} = \vec{T}_1 + \vec{T}_2' + \vec{T}_2'' = -\vec{n} \wedge [-\vec{V}_1 + (\vec{V}_2'(M_e) - \vec{V}_2'(M_i))$$

$$+ (\vec{V}_2''(M_e) - \vec{V}_2''(M_i))] = -\vec{n} \wedge [(\vec{V}_2' + \vec{V}_2'')_{M_e, t} - \vec{V}_E(M, t)], \quad (M \varepsilon \Sigma)$$

(6.18)

On Σ_f, the vortex $\frac{\vec{T}_f(M_f; t)}{\varepsilon}$ is determined by

$$\vec{T}_f(M_f, t) = -\vec{n}_{f_{M_f}} \wedge [\vec{V}(M_{f_2}(\sigma), t) - \vec{V}(M_{f_1}(\sigma), t)] \qquad (6.19)$$

Note that hitherto no assumption had been made concerning the smallness of the motion, except for the fixity of $\mathcal{B}$ on Σ. As in the steady case, the above equations do not entirely determine the fluid motion. That determination also requires the generalized Kutta condition to be taken into account.

The Generalized Kutta Condition

Whether the relative motion is steady or not, the continuity of the fluid motion requires that the three vortex filaments L_{b_1}, L_{b_2}, and L_f be orthogonal to $\mathcal{B}$ (see Chapter 4). We have also seen in Chapter 4 that if the relative motion is steady, this entails the equivalent two formulas Equations (4.21a) and 4.21b) and leads to Equation (4.22).

Let us rewrite (4.22) by taking into consideration the fact that the motion of the body is no longer uniform.

Let, as above, G be the resolvent kernel of Equation (4.22) in Chapter 4. We have:

$$\iint_{\Sigma} \frac{\partial}{\partial \nu_{B(\sigma)}} G\,(B(\sigma),\, M)\ \Phi_f(M,\, t)\ d\Sigma(M)$$

$$= \vec{\nu}_{B(\sigma)} \cdot [\vec{V}'(B_i(\sigma),\, t) - \vec{V}_E(B(\sigma),\, t)] + \iint_{\Sigma} \frac{\partial}{\partial \nu_{B(\sigma)}} G\,(B(\sigma),\, t)\ \Phi'(M_i,\, t) d\Sigma(M) \tag{6.20}$$

On the left-hand side, we have:

$$\begin{cases} \Phi_{f(M,\, t)} = \dfrac{-1}{4\pi} \displaystyle\iint_{\Sigma_{f(t)}} \Gamma_f(\sigma',\, \tau)\ \frac{\partial}{\partial n_f}\ \frac{1}{MM_{f(\sigma)}}\ d\Sigma_f(M(\sigma')), \\ \overrightarrow{B_f(\sigma')M_f(\sigma')} = \displaystyle\int_{\tau}^{t} \vec{V}_R(P(\sigma',\, \tau),\, t')\ dt' \end{cases} \tag{6.21}$$

The right-hand side is of the form:

$$\sum_{j=1}^{6} u_j(t)\ F_j(\sigma) \tag{6.22}$$

where

$$u_1 = c,\ u_2 = \cdots\cdots = u_6 \equiv 0$$

when the motion of the body reduces to a uniform translation of speed c.

Since we are dealing with a small motion about a translation, we may neglect the variation of $\vec{V}_E$ with t provided that we assume $t \gg t_0$. We therefore have:

$$t_o = -\infty\ \overrightarrow{B(\sigma')M_f(\sigma')} = -\,c(t - \tau)$$

Let $x = -X(\sigma)$, $y = Y(\sigma)$, $\frac{dY(\sigma)}{d\sigma} = \eta(\sigma)$be the characteristics of $\mathcal{B}$. By integrating first on Σ_f, Equation (6.20) becomes:

$$-\frac{c}{4\pi}\int_{-\ell}^{+\ell} d\sigma'\eta(\sigma')\int_{-\infty}^{t}\Gamma_f(\sigma', \tau)\, d\tau\left[\iint_\Sigma \frac{\partial}{\partial\nu_{B(\sigma')}} G\,(B(\sigma), M)\,\frac{\partial}{\partial n_f}\,\frac{1}{MM_f(\sigma')}\,d\Sigma(M)\right]$$

$$= c\, F_1(\sigma) + \sum_{j=1}^{6} u_j(t)\, F_j(\sigma) \tag{6.23}$$

Here the six quantities u_1, u_2, u_3, $u_4\,\frac{L}{2}$, $u_5\,\frac{L}{2}$, and $u_6\,\frac{L}{2}$ are small in comparison with c.

Let us put

$$K(\sigma, \sigma'; t - \tau) \tag{6.24}$$

the expression in square brackets in Equation (6.23).

When $u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 0$, that is, when the motion of the body is a uniform translation, (6.24) reduces to

$$-\frac{c}{4\pi}\int_{-\ell}^{+\ell} d\sigma'\eta(\sigma')\int_{-\infty}^{t} K(\sigma, \sigma'; t - \tau)\,\Gamma_f(\sigma', \tau)\, d\tau = c\, F_1(\sigma)$$

By changing τ into $t - \tau'$ and writing τ for τ', we obtain:

$$-\frac{c}{4\pi}\int_{-\ell}^{+\ell} d\sigma'\eta(\sigma')\int_{0}^{\infty} K(\sigma, \sigma'; \tau)\,\Gamma_f(\sigma', t - \tau)\, d\tau = c\, F_1(\sigma)$$

and the limiting function $\Gamma_f(\sigma')$ for $t = \infty$ is solution of the equation

$$- \frac{c}{4\pi} \int_{-\ell}^{+\ell} \eta(\sigma') \left[\int_{0}^{\infty} K(\sigma, \sigma'; \tau)\, d\tau \right] \Gamma_f(\sigma')\, d\sigma' = c\, F_1(\sigma) \tag{6.25}$$

Let us suppose that

$$u_j(t) \equiv 0 \quad \text{except for} \quad j = m \tag{6.26}$$

For the corresponding function $\delta\Gamma_{f_m}$ we obtain

$$- \frac{c}{4\pi} \int_{-\ell}^{+\ell} d\sigma'\, \eta(\sigma') \int_{0}^{\infty} K(\sigma, \sigma'; \tau)\, \delta\Gamma_{f_m}(\sigma', t - \tau)\, d\tau = u_m(t)\, F_m(\sigma) \tag{6.27}$$

Such an equation can be studied by means of either the Laplace transform or the Fourier transform. Let us use the second method. We put:

$$u_m(t) = \int_{-\infty}^{\infty} \exp(i2\pi\nu t)\, \hat{u}_m(\nu)\, d\nu$$

$$\delta\Gamma_{f_m}(\sigma', t) = \int_{-\infty}^{+\infty} \exp(i2\pi\nu t)\, \delta\hat{\Gamma}_{f_m}(\sigma', \nu)\, d\nu \tag{6.28}$$

$\delta\hat{\Gamma}_{f_m}$ and $\hat{u}_m$ are the Fourier transforms of $\delta\Gamma_{f_m}$ and u_m. Let

$$\hat{K}(\sigma, \sigma'; \nu) = \int_{0}^{\infty} \exp(-i2\pi\nu\tau)\, K(\sigma, \sigma'; \tau)\, d\tau \tag{6.29}$$

be the Fourier transform of a function zero for $t < 0$ and equal to K for $t > 0$.

By substituting (6.29) into (6.28), we obtain the integral equation:

$$-\frac{c}{4\pi}\int_{-\ell}^{+\ell} \eta(\sigma')\ \hat{K}(\sigma,\ \sigma';\ \nu)\ \delta\hat{\Gamma}_{f_m}(\sigma',\ \nu)\ d\sigma' = \hat{u}_m(\nu)\ F_m(\sigma) \tag{6.30}$$

This is a Fredholm equation of the first kind. For physical reasons, we assume that there exists one, and only one, solution. In this solution $\delta\hat{F}_m$, the variables σ and ν cannot be separable because the kernel $\hat{K}$ depends on σ and σ' simultaneously. If u_m were a periodic function of t, $\delta\Gamma_m(\sigma,\ t)$ would be periodic too, but both the amplitude and the phase would depend on σ.

The general solution of (6.20) is thus

$$\delta\Gamma_f(\sigma,\ t) = \sum_{j=1}^{6}\int_{-\infty}^{+\infty} \exp\ (i2\pi\nu t)\cdot\delta\hat{\Gamma}_{f_j}(\sigma,\ \nu)\ d\nu \tag{6.31}$$

with δF_{f_j} corresponding to the component u_j of $(\vec{V}_E,\ \vec{\Omega}_E)$.

Hydrodynamic Forces Exerted on the Body in a Small Motion of the First Kind

Recall that the velocity potential on Σ_e is defined by

$$\Phi(M_e,\ t) = \Phi'(M_i,\ t) + \mu'(M,\ t) + \mu''(M,\ t) \tag{6.32}$$

$$\Phi(M_e,\ t) = \left\{\sum_{j=1}^{6} u_j(t)\left[\Phi_j'(M_i,\ t) + \iint_\Sigma G\ (M,\ P)\ \Phi_j(P_i,\ t)\ d\Sigma(P)\right]\right\} - \iint_\Sigma G\ (M,\ P)\ \Phi_f(P,\ t)\ d\Sigma(P) \tag{6.33}$$

In (6.33), the expression in braces would exist alone if no free vortex sheet were shed by the body.

According to the Euler equation, the hydrodynamic pressure exerted on the body in the moving reference system is given by

$$\frac{1}{\rho} P_d(M_e, t) = - \left(\frac{\partial \Phi}{\partial t}\right)_{M'_e, t} + \frac{1}{2} (\Omega_E^2 r^2)_{M, t} - \frac{1}{2} V_R^2(M_e, t) \tag{6.34}$$

$$= - \left(\frac{\partial \Phi}{\partial t}\right)_{M'_e, t} + \frac{1}{2} (\Omega_E^2 r^2)_{M, t} - \frac{1}{2} (\vec{T}_1 + \vec{T}'_2 + \vec{T}''_2)^2 \tag{6.35}$$

Now recall that on the surface of the body there are three local systems of axes:

$$(\vec{n}, \vec{\theta}, \vec{\tau}), (\vec{n}, \vec{\theta}'_2, \vec{\tau}'_2), (\vec{n}, \vec{\theta}''_2, \vec{\tau}''_2)$$

$\vec{\tau}_1$, $\vec{\tau}'_2$, $\vec{\tau}''_2$ are in the directions of $\vec{T}$, $\vec{T}'_2$, $\vec{T}''_2$, respectively. On Σ, we also consider the curves C'_2, C''_2 each of which is tangent to the corresponding vector $\vec{\theta}'_2$ or $\vec{\theta}''_2$. Their arcs are in the directions of these vectors.

We have

$$\vec{T}_1 = \vec{n} \wedge \vec{V}_1, \quad \vec{T}'_2 = \frac{\partial \mu'}{\partial \sigma'_2} \vec{\tau}'_2, \quad \vec{T}''_2 = \frac{\partial \mu''}{\partial \sigma''_2} \vec{\tau}''_2 \tag{6.36}$$

The hydrodynamic system of forces due to the added masses is the system

$$S_{in} = [p_{d\ in}\, \vec{n}\, d\Sigma] = \left\{- \rho \sum_{j=1}^{6} \dot{u}_j(t) \left[\Phi_j(M_i) + \mu'_j(M)\right] \vec{n}_M\, d\Sigma(M)\right\} \tag{6.37}$$

The system due to the apparent added masses is

$$S'_{in} = [p'_{d\ in}\, \vec{n}\, d\Sigma] = \left[- \rho \left\{\iint_\Sigma G(M, P) \frac{\partial}{\partial t} \Phi_f(P, t)\, d\Sigma(P)\right\} \vec{n}\, d\Sigma(M)\right] \tag{6.38}$$

The system of the quasi-steady forces is

$$S_{q.s} = - \vec{F}_c(t) - \left\{ \frac{1}{2} \rho \left[\vec{T}(M, t) \wedge \vec{V}_R(M_e, t) \right] \; d\Sigma(M) \right\}$$

$$= - \vec{F}_c(t) - \left[\frac{1}{2} \rho \, (\vec{T}_1 + \vec{T}'_2 + \vec{T}''_2{}_{M,\,t})^2 \vec{n}_M \, d\Sigma(M) \right] \tag{6.39}$$

In this formula for determining the quasi-steady system of hydrodynamic forces exerted on the body, $\vec{T}''_2(M, t)$ depends on the history of the motion since the contribution to Φ from Φ_f is

$$\left.\begin{aligned}
&\Phi''_2 = \frac{-c}{4\pi} \int_{-\ell}^{\ell} d\sigma' \tau(\sigma') \int_{-\infty}^{t} \delta\Gamma_f(\sigma', \tau) \, K_1(M, \sigma', t - \tau) \, d\tau, \text{ with} \\
&K_1(M, \sigma', t - \tau) = \iint_{\Sigma} G(M, P) \, \frac{\partial}{\partial n_f} \frac{1}{MP(\sigma')} d\Sigma(P) \text{ and} \\
&- \frac{c}{4\pi} \int_{-\ell}^{+\ell} d\sigma' \tau(\sigma') \int_{-\infty}^{t} \delta\Gamma_f(\sigma', \tau) \, K(\sigma, \sigma', t - \tau) \, d\tau = \sum_j u_j(t) \, F_j(\sigma)
\end{aligned}\right\} \tag{6.40}$$

In practice, what is called the quasi-steady system of forces is that derived from

$$\left.\begin{aligned}
&\Phi''_2 = - \frac{c}{4\pi} \int_{-\ell}^{+\ell} d\sigma' \tau(\sigma') \left[\int_{-\infty}^{t} K_1(M, \sigma', t - \tau) \, d\tau \right] \delta\Gamma_f(\sigma') \text{ with} \\
&- \frac{c}{4\pi} \int_{-\ell}^{\ell} d\sigma' \tau(\sigma') \left[\int_{-\infty}^{t} K(\sigma, \sigma', t - \tau) \, d\tau \right] \delta\Gamma_f(\sigma') = \sum_j u_j(t_0) F_j(\sigma)
\end{aligned}\right\} \tag{6.41}$$

where t_0 in the last equation represents the present time. The quasi-steady system of forces is therefore those forces which would be reached at $t = \infty$ with $u_j(t) = u_j(t_0)$ in the interval $[t_0, + \infty)$.

Let $\frac{1}{\varepsilon}\vec{T}''^{0}_{2}$ denote the bound vortex corresponding to Equation (6.41) and let $S^{0}_{q.s.}$ denote the quasi-steady system which would be derived from the distribution $\left(\Sigma, \frac{\vec{T}''^{0}_{2}}{\varepsilon}\right)$. We have

$$\Delta S_{q.s.} = S^{0}_{q.s.} - S_{q.s.} = -\frac{1}{2}\rho\left\{\left(\vec{T}''^{0}_{2} - \vec{T}''_{2}\right)\left[\vec{T}_{1} + \vec{T}_{2} + \frac{1}{2}\left(\vec{T}''_{2} + \vec{T}''^{0}_{2}\right)\right]\vec{n}_{M}\, d\Sigma(M)\right\} \tag{6.42}$$

This difference is the "deficiency" due to the difference $\delta\Gamma_{f}(\sigma')-\delta\Gamma_{f}(\sigma', t)$.

For the sake of simplicity, we very often accept the questionable assumption according to which the accelerations $\frac{d\vec{V}_{E}(0)}{dt}$ and $\frac{d\vec{\Omega}_{E}}{dt}$ are small enough for

$$S'_{in} + \Delta S_{q.s} \tag{6.43}$$

to be practically negligible.

SMALL MOTIONS OF THE SECOND KIND

Example of a Double Model: Structure of the Vortex Distribution

The planes of symmetry of the body are the (z, x)-plane which is "vertical" and the (x, y)-plane which is "horizontal." The contour C intersected in the hull by the (z, x)-plane is described in the direction *from Oz toward Ox, Oz being vertical upward.* It consists of the stem E_1EE_0, of the stern post S_0SS_1 and of two horizontal segments of straight lines E_0S_0 and S_1E_1 in the planes z = -H and z = +H, respectively. Let λ be the curvilinear abscissa on C; its origin is at E. The abscissas of $EE_0E_1S_0$, S_1 and S are as follows:

$$\lambda(S) = \pm\Lambda,\ \lambda(S_1) = -\lambda(S_0) = \lambda_1,\ \lambda(E_1) = -\lambda(E_0) = -\lambda_0$$

The "mean" velocity of the origin 0 of the moving system S of axes is in the direction from S to E. The cartesian abscissas of S_0 and S_1 and of E_0 and E_1 are $-\ell$ and $+\ell$, respectively.

The contour C is thus identical with that of the double models considered in Chapters 4 and 5. The aspect ratio is $\frac{H}{\ell}$ or $\frac{H}{L}$. Here L is the length SE and the aspect ratio is <u>very</u> small.

In the small motions of the double model, the components of $\vec{V}_E(0)$ on the moving axes are $c + u_1$, u_2, u_3 (c being a constant) and the components of $\vec{\Omega}_E$ are u_4, u_5, u_6. The squares of u_1, u_2, u_3 and $u_4\ell$, $u_5\ell$, $u_6\ell$ are supposed to be negligibly small in comparison with c^2. If these small motions were totally in the horizontal plane, u_3, u_4, u_5 would be zero.

It is assumed that no free vortex is shed from the stem. All the free vortex filaments start from or arrive at points on $E_0S_0S_1E_1$. Their ends B_0 and B_1 are mirror images of each other with respect to the horizontal (x, y)-plane. The point B_0 is located on the arc E_0S_0S and the point B_1 on the arc SS_1E_1.

A free vortex filament L_f starting from B_1 and arriving at B_0 at t has at t + dt two short "horizontal" branches B_1M_{f1} and $M_{f0}B_0$ such that

$$\overrightarrow{B_1M_{f_1}} = \int_t^{t + dt} \vec{V}_R(P_1(t), t')\, dt', \quad \overrightarrow{B_0M_{f_0}} = \int_t^{t + dt} \vec{V}_R(P_0(t), t')\, dt',$$

P_0 and P_1 being the fluid points which were at B_0 and B_1 at t. The fluid points on $M_{f1}\,M_{f0}$ were at t to port, and therefore, the bound part L_b included in the same vortex filament L as L_f was necessarily located on an arc B_1B_0 located on the port side Σ_1 of Σ. On the contrary, if at t, L_f started from B_0 and arrived at B_1, the arc $M_{f0}M_f$ of L_f would necessarily be in the region y < 0 and L_f would be closed at t + dt by a bound vorted filament located on the starboard side Σ_2 of Σ.

This explains why, anticipating this circumstance, we have assumed in Chapter IV, that no vortex filament including a free vortex filament is divided on Σ into two parts, one of them located on Σ_1 and the other one on Σ_2.

If all the u_j's were the constant zero, no free vortex would be shed. The two arcs of bound vortex filaments joining B_0 to B_1 on Σ_1 and B_1 to B_0 on Σ_2 would be symmetric to each other with respect to the (z, x)-plane and their intensities $d_\lambda\ \Gamma_1(\lambda)$, $d_\lambda\ \Gamma_2(\lambda)$ would be equal. This is no longer true in the case of the small motions under consideration.

Let $\{t_n\}$ denote the sequence of times t_n at which $d_\lambda\ \Gamma_1(\lambda, t)$ and $d_\lambda\ \Gamma_2(\lambda, t)$ are equal to each other. Suppose that the y-component of the incident velocities at $B_1\ (-\lambda)$ and $B_0\ (-\lambda)$ during the interval (t_n, t_{n+1}) is positive. Then at any $t\ \varepsilon(t_n, t_{n+1})$, the intensity $d_\lambda\ \Gamma_1(\lambda, t)$ of the bound vortex $C_1'(\lambda, t)$ joining $B_0\ (\lambda)$ to $B_1\ (-\lambda)$ is greater than the intensity $d_\lambda\ \Gamma_2(\lambda, t)$ of the bound vortex $C_2'(\lambda, t)$ joining $B_1\ (-\lambda)$ to $B_0\ (\lambda)$ on Σ_2. The difference $d_t d_\lambda[\Gamma_1(\lambda, t) - \Gamma_2(\lambda, t)]$ increases during the first part (t_n, t_n') of $t_n, t_{n+1})$ and decreases during the second part (t_n', t_{n+1}).

If $t\in(t_n, t_n')$, the positive variation $d_\tau d_\lambda[(\Gamma_1, \tau) - \Gamma_2(\lambda, \tau)]$ of $d_\lambda[\Gamma_1(\lambda, \tau) - \Gamma_2(\lambda, \tau)]$ determines the shedding of a free vortex filament $B_1\ (-\lambda)\ M_{f_1}(\lambda, \tau;\ \tau+d\tau)\ M_f(\lambda, \tau;\ \tau+d\tau)\ M_{f_0}(\lambda, \tau;\ \tau+d\tau)\ B_0(\lambda)$ closed by the arc $C_1'(\lambda, \tau + d\tau)$ joining $B_0(\lambda)$ to $B_1\ (-\lambda)$ at $\tau + d\tau$ on Σ_1. The fluid points which were at τ at $B_1\ (-\lambda)$ and $B_0\ (\lambda)$ are at $M_{f_1}\ (\lambda, \tau;\ \tau+d\tau)$ and $M_{f_0}\ (\lambda, \tau;\ \tau+d\tau)$ at $\tau+d\tau$, the short arcs $B_1\ (-\lambda)\ M_{f_1}(\lambda, \tau;\ \tau+d\tau)$ and $B_0\ (\lambda)\ M_{f_0}\ (\lambda, \tau;\ \tau+d\tau)$ are approximately horizontal. We have

$$\left.\begin{aligned} \overline{B_1(-\lambda)\ M_{f_1}(\lambda,\ \tau;\ \tau+d\tau)} &= \vec{V}_i(-\lambda, \tau)\ d\tau \\ \overline{B_0(\lambda)\ M_{f_0}(\lambda,\ \tau;\ \tau+d\tau)} &= \vec{V}_i(\lambda,\ \tau)\ d\tau \end{aligned}\right\} \cong -\vec{V}_E(\lambda,\ \tau)\ d\tau$$

$\vec{V}_i(-\lambda, \tau)$ and $\vec{V}_i(\lambda, \tau)$ are the incident velocities at $B_1(-\lambda)$ and $B_0(\lambda)$ at time τ. At $t > \tau$, these two fluid points are at $M_{f_0}(\lambda, \tau;\ t)$, $M_{f_1}(\lambda, \tau;\ t)$, respectively, and the arc $M_{f_0}(\lambda, \tau;\ \tau+d\tau)\ M_f(\lambda, \tau, \tau+d\tau)\ M_{f_0}(\lambda, \tau;\ \tau+d\tau)$ is at t an arc of free vortex filament whose position and intensity are $M_{f_1}(\lambda, \tau;\ t)\ M_f(\lambda, \tau;\ t)\ M_{f_0}(\lambda, \tau;\ t)$ and $d_\tau\ d_\lambda\ [\Gamma_1(\lambda, \tau) - \Gamma_2(\lambda, \tau)]$, respectively.

If $\tau\varepsilon(t_n', t_{n+1})$, the free vortex filament shed during the time interval $(\tau, \tau+d\tau)$ is analogous to the preceding one but its intensity $d_\tau\, d_\lambda [\Gamma_1(\lambda, \tau) - \Gamma_2(\lambda, \tau)]$ along the arc $B_1(-\lambda)\ M_{f_1}(\lambda, \tau; \tau+d\tau)\ M_f(\lambda, \tau; \tau+d\tau)\ M_{f_0}(\lambda, \tau; \tau+d\tau)\ B_0(\lambda)\ B_1(-\lambda)$ is now negative.

Whatever t may be inside the interval (t_n, t_{n+1}), any point on the line $B_1(-\lambda)\ M_{f_1}(\lambda, t_n; t)$ may be considered the position at t of a fluid point which was at $B_1(-\lambda)$ at a certain time $\tau\varepsilon(t_n, t)$. Thus it can be denoted $M_{f1}\ (\lambda, \tau; t)$. The interval $t - \tau$ is roughly proportional to the distance from $B_1\ (-\lambda)$. The intensity at $M_{f1}\ (\lambda, \tau; t)$ of the free vortex filament whose support is the line $B_1\ (-\lambda)\ M_{f1}\ (\lambda, t_n; t)$ is as

$$\int_{t_n}^{\tau} dt'\ d_\lambda\ [\Gamma_1(\lambda, t') - \Gamma_2(\lambda, t')] = d_\lambda\ [\Gamma_1(\lambda, \tau) - \Gamma_2(\lambda, \tau)]$$

since $d_\lambda[\Gamma_1(\lambda, t_n) - \Gamma_2(\lambda, t_n)] = 0$. The points $M_{f_0}(\lambda, \tau; t)$ are defined in a similar manner on the line $B_{f_0}M_{f_0}(\lambda, t_n; t)$. Along the arc $M_{f_1}(\lambda, \tau; t)\ M_f(\lambda, \tau; t)\ M_{f_0}(\lambda, \tau; t)$, the intensity $d_\tau\ d_\lambda[\Gamma_1(\lambda, \tau) - \Gamma_2(\lambda, \tau)]$ of the free vortex filament is positive if $\tau\varepsilon(t_n, t_n')$ and negative if $\tau\varepsilon(t_n', t_{n+1})$.

Let $S_1(\lambda, t_n; t)$ denote the surface generated by the arc $M_{f_1}\ M_f\ M_{f_0}$ during the time interval $[t_n, t)$. When t reaches time t_{n+1}, the intensity of the free vortex $M_{f_1}(\lambda, t - 0; t)\ M_f(\lambda, t - 0; t)\ M_{f_0}(\lambda, t - 0; t)$ vanishes and the surface $S_1(\lambda, t_n; t_{n+1})$ is completely free for $t > t_{n+1}$; it moves with the fluid. For $t'\ \varepsilon(t_{n+1}, t_{n+2})$, there exists a surface $S_2(\lambda, t_{n+1}; t')$ analogous to $S_1(\lambda, t_n; t)$ but starting from the arc $C_2'(\lambda, t')$ which is the support at t' of the bound vortex filament joining B_1 to B_0 on Σ_2.

Figure 13 indicates the relative positions of the body and of the surface $S_1(\lambda; t_n, t)$ at $t\ \varepsilon(t_n, t_{n+1})$; Figure 14 shows the relative positions of the body and of the two surfaces $S_1(\lambda, t_n; t_{n+1})$, $S_2(\lambda, t_{n+1}, t')$ at $t'\ \varepsilon(t_{n+1}, t_{n+2})$.

If we are dealing with a "pure" swaying motion, all the incident velocities at $B_1(-\lambda)$ and $B_0(\lambda)$ have positive y-components during the even intervals (t_{2p}, t_{2p+1}), and negative y-components during the odd intervals (t_{2p+1}, t_{2p+2}),... Therefore, if, for instance, $t \varepsilon (t_{2n+1}, t_{2n+2})$, all the surfaces $S_1(\lambda, t_{2p}; t_{2p+1})$ and $S_2(\lambda, t_{2p+1}; t_{2p+2})$ are free for $t_{2p} < t_{2n+1}$; only the surfaces $S_2(\lambda, t_{2n+1}, t_{2n+2})$ are in contact with the body along the arcs $C_2'(\lambda, t)$.

In the case of a "pure" yawing motion, $\vec{V}_E(0)$ is tangent to the (z, x)-plane at any t. At a given time t, the y-component of the incident velocities has a certain sign for $\lambda > \lambda(0)$ and the opposite sign for $\lambda < \lambda(0)$. These signs depend on that of the z-component of $\vec{\Omega}_E$. If the latter is positive, the surfaces S_2 corresponding to the interval $\lambda > \lambda(0)$ and the surface S_1 corresponding to the interval $\lambda < \lambda(0)$ are in contact with the body along arcs $C_2'(\lambda, t)$ and $C_1'(\lambda, t)$, respectively.

In both cases, the vortex wake is three-dimensional. The small motions of the second kind are thus much more intricate than the small motions of the first kind.

In such a motion that v_E becomes a constant different from zero for $t > t_0$, while w_E and $\vec{\Omega}_E$ vanish, then the surfaces S_1 and S_2 are carried by the flow at infinity downstream from the body, and in the limiting steady motion, the vortex wake reduces to the surface $\Sigma_{f_1} + \Sigma_{f_0} + \Sigma_f'$ considered in Chapter 4.

Calculation of the Velocity Field

The total velocity $\vec{V}$ due to the motion of the body is defined inside D_e and inside D_i by a vortex distribution $\mathcal{D}$. It is expressed by formulas analogous to those of the preceding sections.

We may write

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2' + \mathcal{D}_2''$$

We have first

$$\mathcal{D}_1 = \left(\Sigma, \frac{\vec{f}_1}{\varepsilon}\right) + \left(D_i, 2\vec{\Omega}_E\right)$$

The velocity $\vec{V}_1$ generated by $\mathcal{D}_1$ is zero inside D_e. Within D_i, we have

$$\vec{V}_E - \vec{V}_1 = \nabla\Phi'$$

We know from discussions of the effective determination of the vortex distribution (Chapter 3) how to calculate $\vec{T}_1$ and thus Φ'. A second part of $\mathcal{D}$ consists of vortices $\frac{1}{\varepsilon}\vec{T}_2'$ distributed on Σ. The corresponding vortex distribution

$$\mathcal{D}_2' = \left(\Sigma, \frac{1}{\varepsilon}\vec{T}_2'\right)$$

is equivalent to a normal doublet distribution on Σ:

$$\mathcal{D}_2' \sim (\Sigma, \mu' \vec{n})$$

and the density μ' is selected so that the velocity potential

$$\Phi_2' = \frac{-1}{4\pi} \iint_{\Sigma} \mu'(M) \frac{\partial}{\partial n_{M'}} \frac{1}{MM'} \, d\Sigma(M')$$

satisfies the boundary condition $\Phi_2' \equiv \Phi$ on Σ_i.

The velocity $\vec{V}_1 + \nabla\Phi_2'$ thus satisfies the condition

$$\vec{V}_1 + \nabla\Phi_2' = \vec{V}_1 + (\vec{V}_E - \vec{V}_1) = \vec{V}_E \text{ within } D_i$$

This shows that

$$\vec{n} \cdot (\vec{V}_1 + \vec{V}_2') \equiv \vec{n} \cdot \vec{V}_E \text{ on } \Sigma_e \text{ as well as on } \Sigma_i$$

Consequently, the vortex distribution $\mathcal{D}_1 + \mathcal{D}_2'$ is that with which we should deal if no free vortex were shed by the body.

In the case of motions of the first kind, the remaining vortex distribution $\mathcal{D}_2''$ consisted of vortex filaments L, each L being located partly on the hull surface Σ and partly on the free vortex sheet Σ_f. It has been shown that $\mathcal{D}_2''$ is equivalent to a normal doublet distribution on Σ and on Σ_f. Symbolically, we had

$$\mathcal{D}_2'' \sim (\Sigma, \mu' \vec{n}) + (\Sigma_f, \mu_f \vec{n}_f).$$

The velocity $(\vec{V} - \vec{V}_2')$ generated by $\mathcal{D}_2''$ was expressed as:

$$\begin{cases} \nabla\Phi'' + \nabla\Phi_f \text{ with} \\ \Phi'' = \dfrac{-1}{4\pi} \displaystyle\iint_{\Sigma} \mu''(M') \frac{\partial}{\partial n_{M'}} \frac{1}{MM'} \, d\Sigma(M') \\ \Phi_f = \dfrac{-1}{4\pi} \displaystyle\iint_{\Sigma} \mu_f(M_f) \frac{\partial}{\partial n_{M_f}} \frac{1}{MM_f} \, d\Sigma_f(M_f) \end{cases}$$

The density μ'' was determined by the condition

$$\Phi''(M_i, t) \equiv -\Phi_f(M_i, t) \text{ on } \Sigma_i$$

and μ_f was the solution of an integral equation which states that the pressure p on Σ is continuous through the line $\mathcal{B}$ from which the free vortex filaments are shed.

The same principle applies in the case of motions of the second kind but with some modifications because the vortex wake is now three-dimensional. Let us suppose, for instance, that $t \in (t_{2n}, t_{2n+1})$.

The free vortex filaments shed at times t prior to t_{2n} are located within domains W_{2p}, W_{2p+1}; each domain W_{2p} contains all the free vortices shed during the time interval $[t_{2p}, t_{2p+1})$, and each domain W_{2p+1} contains all the free vortices shed during the time interval $[t_{2p+1}, t_{2p+2})$. The domain W_{2p} is generated by the surfaces $S(\lambda, t_{2p+1}, t_{2p+2})$. In a pure swaying motion, all the surfaces $S(\lambda, 2p, 2p+1)$ are of type S_1 and all the

surfaces $S(\lambda, t_{2p+1}, t_{2p+2})$ are of type S_2. In a pure yawing motion, each domain W_{2p}, W_{2p+1} is subdivided into two domains; for instance:

$$W_{2p} = W'_{2p} + W''_{2p}$$

where W'_{2p} is generated only by surfaces S_1 and W''_{2p} only by surfaces S_2. The domains W_{2p} and W_{2p+1} are disjoint.

Furthermore, the vortex wake contains a domain W^t_{2n} whose structure differs from that of domain W_{2p} or W_{2p+1}. The explanation is that a surface such as $S_1(\lambda, t_{2n}; t)$ includes an arc of bound vortex filament whose intensity is $d_\lambda[(\Gamma_1(\lambda, t) - \Gamma_2(\lambda, t)]$ and whose support is $C'_1(\lambda, t)$; in an analogous manner, if W^t_{2n} also contains surfaces $S_2(\lambda', t_{2n}; t)$, each of these surfaces includes an arc of vortex filament of intensity $d_{\lambda'}[\Gamma_2(\lambda', t) - \Gamma_1(\lambda', t)]$ whose support is the arc $C'_2(\lambda', t)$.

The velocity induced by the vorticity distributed inside a domain W_{2p} is given by

$$\vec{V}_{2p}(M, t) = \text{curl}\ \frac{1}{4\pi} \iiint_{W_{2p}} \frac{\vec{\omega}_f(M', t)}{MM'}\ dW_{2p}(M')$$

with $\vec{\omega}_f = \frac{1}{\varepsilon}\vec{T}_f$ if M' belongs to the cover $(\Sigma_{f_1} + \Sigma_{f_0} + \Sigma'_f)_{2p}$ of W_{2p}.

Similarly, we have

$$\vec{V}_{2p+1}(M, t) = \text{curl}\ \frac{1}{4\pi} \iiint_{W_{2p+1}} \frac{\vec{\omega}_f(M', t)}{MM'}\ dW_{2p+1}(M')$$

Let us now consider the velocity $\vec{V}^t_{2n}$ induced by the vortex filaments located inside W^t_{2n}. In a pure swaying motion, W^t_{2n} is adjacent to part Σ'_1 of the portside Σ_1 of the hull downstream from the arc $C'_1(\lambda_0, t)$. In a pure yawing motion, W^t_{2n} is subdivided into two parts W'^t_{2n} and W''^t_{2n} respectively adjacent to a part Σ'_1 of Σ_1 and to a part Σ'_2 of Σ_2. The arcs of vortex

filaments of intensity $d_\lambda[\Gamma_1(\lambda, t) - \Gamma_2(\lambda, t)]$ located on Σ_1' and those of the vortex filament of intensity $d_\lambda[\Gamma_2(\lambda, t) - \Gamma_1(\lambda, t)]$ located on Σ_2' form two vortex sheets. Consequently V_{2n}^t is discontinuous through Σ_1' and through Σ_2'. Its jump is tangent to Σ_1' or to Σ_2' and zero through $\Sigma - (\Sigma_1' + \Sigma_2')$.

The velocity $\vec{V}_{2n}^t$ is given by

$$\vec{V}_{2n}^t = \text{curl}\ \frac{1}{4\pi} \iiint_{W_{2n}^t} \frac{\vec{\omega}_f(M', t)}{MM'}\ dW_{2n}^t(M')$$

with $\vec{\omega}_f = \frac{1}{\varepsilon}\vec{T}_f$ on the cover $(\Sigma_{f_1}^t + \Sigma_{f_0}^t + \Sigma_f^t + \Sigma_1' + \Sigma_2')$ of W_{2n}^t.

The resulting velocity

$$\vec{V}_w(M, t) = \sum_{r<n} [\vec{V}_{2r}(M, t) + \vec{V}_{2r+1}(M, t)] + \vec{V}_{2n}^t(M)$$

is irrotational within D_i. We may write:

$$\vec{V}_w(M, t) = \nabla\Phi_w(M, t) \text{ when } M \in D_i$$

Obviously $\vec{V}_w(M_e, t)$ is not tangent to Σ_e. Consequently, the vortex distribution $\mathcal{D}_2''$ contains closed bound vortex filaments located on Σ. They induce inside D_i a velocity $\vec{V}_2''$ which is irrotational everywhere except on Σ and which meets the requirement

$$\vec{V}_2''(M, t) = -\vec{V}_w(M, t) \text{ at every point } M \in D_i$$

This complementary vortex distribution is equivalent to a normal doublet distribution on Σ. Let Φ_2'' denote the potential generated by this doublet distribution. We may write

$$\vec{V}_2'' = \nabla\Phi_2'', \text{ with } \Phi_2''(M, t) = \frac{-1}{4\pi} \iint_\Sigma \mu''(M', t) \frac{\partial}{\partial n_{M'}} \frac{1}{MM'}\ d\Sigma(M')$$

and μ" is determined by the regular Fredholm equation which states that

$$\Phi''_2 \equiv - \Phi_w \text{ at every point } M_i \text{ on } \Sigma_i$$

The corresponding bound vortex $\frac{1}{\varepsilon}\vec{T}''_2$ is

$$\frac{1}{\varepsilon}\vec{T}''_2(M, t) = - \vec{n}_M \wedge [\nabla\Phi''_2(M_e, t) - \nabla\Phi''_2(M_i, t)]$$

The total vortex distribution $\mathcal{D}''_2$ may symbolically be written in the form:

$$\mathcal{D}''_2 = \left(\Sigma, \frac{1}{\varepsilon}\vec{T}''_2\right) + \sum_{r<n} [(W_{2r}, \vec{\omega}_f) + (W_{2r+1}, \vec{\omega}_f)] + (W^t_{2n}, \vec{\omega}_f)$$

The total bound vortex on Σ is

$$\frac{1}{\varepsilon}\vec{T} = \frac{1}{\varepsilon}(\vec{T}_1 + \vec{T}'_2 + \vec{T}''_2) + \begin{cases} \frac{1}{\varepsilon}\vec{T}'_{f_1} \text{ on } \Sigma'_1 \\ \\ \frac{1}{\varepsilon}\vec{T}'_{f_2} \text{ on } \Sigma'_2 \end{cases}$$

where $\vec{T}'_{f1}$ and $\vec{T}'_{f2}$ belong to W'^t_{2n} and W''^t_{2n} respectively. $\vec{T}$ is tangent to C'_1 (λ, t) on Σ_1 and to $C'_2(\lambda, t)$ on Σ_2, and the total intensity of the arc of bound vortex filament which is the support of $\vec{T}$ is $d_\lambda\Gamma_1(\lambda, t)$ on C'_1 and $d_\lambda\Gamma_2(\lambda, t)$ on C'_2.

The Generalized Kutta Condition in Small Motions of the Second Kind

Let $B_0(\lambda)$ be a point on E_0S_0S, and let P, Q denote two points very close to B_0 on $C'_1(\lambda, t)$ and $C'_2(\lambda, t)$ respectively, (see Figure 15). We assume that

$$d_\lambda\Gamma_1(\lambda, t) > d_\lambda\Gamma_2(\lambda, t) \tag{6.44}$$

and, therefore, that B_0 is the end of a free vortex filament L_{f_0} of intensity $d_\lambda[\Gamma_1'(\lambda, t) - \Gamma_2'(\lambda, t)]$. Let P_e, Q_e be two points on Σ_e with $\overrightarrow{P_eP} = \vec{n}_P(0+)$, $\overrightarrow{Q_eQ} = \vec{n}_Q\ (0+)$. The generalized Kutta condition means that

$$\lim \frac{1}{\rho}\left[p_d(P_e, t) - p_d(Q_e, t)\right] = 0 \tag{6.45}$$

when P, Q tend to B_0 (Figure 15). To express this condition more explicitly, consider a point b very close to P_e, $\overrightarrow{b\,P_e}$ being in the direction of L_{f_0}. This point thus belongs to W'^{t}_{2n}. We also consider a point P' on the lower side of Σ_{f_0}, with b P' orthogonal to Σ_{f_0}. The flow is irrotational at P' and at Q_e. The condition (6.45) is thus equivalent to

$$\lim \frac{1}{\rho}\left[p_d(b, t) - p_d(P', t)\right] = 0 \text{ when P, Q tend to } B_0 \tag{6.46}$$

When λ decreases so that B_0 moves in the positive x-direction, $d_\lambda[\Gamma_1'(\lambda, t) - \Gamma_2'(\lambda, t)]$ decreases and vanishes at a certain point $B_0'(\lambda')$. Simultaneously, P' describes an arc $P'P_0'$ and b an arc $b\,b_0'$. Consider the path P consisting of the arcs $P'P_0'$, $P_0'b_0'$, $b_0'b$. According to the Euler equation, we have:

$$\frac{1}{\rho}\left[p_d(b, t) - p_d(P', t)\right] = -\frac{\partial}{\partial t}\int_P \vec{V}\vec{d}s - \int_P (\vec{\omega} \wedge \vec{V}_R)\ ds - \frac{1}{2}\left[V_R^2(b) - V_R^2(P')\right] \tag{6.47}$$

At b, $\vec{\omega}$ is tangent to the arc $M_{f_1}(\lambda, t - 0; t)\ M_{f_0}(\lambda, t - 0; t)$ and thus is nearly parallel to $C_1'(\lambda, t)$. Hence $\vec{\omega} \wedge \vec{V}_R$ is normal to Σ, and when P tends to B_0, it is normal to the arc $b_0'b$. As $\vec{\omega} = 0$ on $P'P_0'$, and $P_0'b_0'$ is infinitely small, we have

$$\int_P (\vec{\omega} \wedge \vec{V}_R)\ .\ \vec{d}s = 0$$

On the other hand, $\int_{P+bP'} \vec{V}\, d\vec{s}$ becomes equal to the flux of $\vec{T}_{f_0}$ through the closed contour $P'P_0'b'bP'$, and we have

$$\int_P \vec{V}\, d\vec{s} = \int_{\lambda_0'}^{\lambda} d_\lambda[\Gamma_1(\lambda, t) - \Gamma_2(\lambda, t)] = \Gamma_1(\lambda, t) - \Gamma_2(\lambda, t)$$

Finally, since $\vec{V}_{R_1}$ and $\vec{V}_{R_2}$ are the relative velocities at B_0 on Σ_1 and on Σ_2, respectively, Equation (6.46) reduces to

$$-\frac{\partial}{\partial t}[\Gamma_1(\lambda, t) - \Gamma_2(\lambda, t)] - \frac{1}{2}[V_{R_1}^2(B_0, t) - V_{R_2}^2(B_0, t)] = 0 \quad (6.48)$$

This is the generalized Kutta condition. It should be compared with the result for the general Kutta condition given in Chapter 4 for bodies in an oblique, uniform translation in the horizontal plane.

CHAPTER 7: REMARKS ON SOME APPLICATIONS TO SHIPS

INTERACTION OF APPENDAGES WITH THE HULL

These lecture notes would be very incomplete if nothing were said about the appendages with which a ship is fitted, and, more particularly, those whose role is to allow changes of heading or depth or to give to the ship good coursekeeping characteristics.

In fact, the first systematic researches on maneuverability began a century or so ago with the testing of isolated rudders. At that time, nothing was known about lifting surface theory and the experiments were therefore very useful. In France, experiments were carried out by Joessel at Indret on the River Loire. His empirical formulas were used for several decades by naval architects as guides for predicting the horizontal components of the hydrodynamic forces exerted on the rudders and for determining the size of the axle, the position and dimensions of the bearings, and the structural design of the stern post.

Experiments on isolated rudders are now obsolete inasmuch as the properties of a given rudder depend considerably on the interaction between rudder and hull. For example, the flow about a rudder located behind a skeg is very different from the flow about the same rudder when there is a large gap between the hull and the leading edge of the rudder (see Figures 16a and 16b). The effect of the rudder is delayed in the first case, but is quasi-immediate in the second. In the first case, the rudder and the skeg form a whole; in the second case, the circulation of the velocity in a circuit surrounding the rudder is established almost instantaneously. The time to establish the circulation around an obstacle is the time it takes the flow to cover the length of the obstacle.

It is known that the moment about the ship CG of the hydrodynamic force on the hull plus rudder is generally greater than the moment due to the contribution from the rudder alone. This effect is greater when the vertical gap between the hull and the upper edge of the rudder is small and the transverse sections of the hull in the region of the rudder are flat. The vicinity of the hull prevents the upper edge tip vortex from shedding and increases the apparent aspect ratio of the rudder. It may even happen that the bound vortices on the rudder are continued on the hull until the free

surface. In this case, the amplification factor may be greater than 2 (Figure 17).

A rudder may give rise to separation. The smaller the aspect ratio is, the greater the angle at which separation occurs. Therefore, the best rudder is not necessarily one with the highest aspect ratio. Furthermore, course stability often requires the addition of a fin ahead of the rudder.

If the rudder is joined to a fixed strut whose length is equal to one-quarter or one-third of the chord of the rudder, the resulting rudder works somewhat like a cambered wing at rudder angles other than zero. This decreases the risk of separation and increases the lift coefficient of the rudder (Figure 18).

On modern submarines, the rudder consists of two parts that are approximately symmetric to each other with respect to the longitudinal axis of the vessel. Clearly the bound vortices on the two parts are connected by bound vortices on the afterbody (Figure 19). In this region, the two parts of the rudder must be regarded as included in the longitudinal contour of the ship. The same is true of the sail of a submarine.

In consideration of the preceding chapters, I do not believe that it is possible to set forth a general rule which can take into account the effect of the appendages on the maneuvering qualities of a given ship. Each case should be considered separately before deciding whether it has to be incorporated into the hull or be regarded as separated from it.

UNSTEADY MOTIONS ABOUT A WING PROFILE

It has been shown in Chapter 4 that by virtue of the generalized Kutta condition, the relative velocity on both sides of a wing with a finite aspect ratio and a finite thickness is tangent to the trailing edge at any point from which a free vortex is shed. From this it follows that the assumption of an almost inviscid fluid necessarily contradicts either the continuity of the motion or the Lagrange theoren. The Lagrange theorem itself is a consequence of the Helmholtz theorem which implies continuity of the motion.

Since the Kutta condition requires that the relative velocity be zero at the trailing edge, there is an analogous contradiction in the case of a

wing profile when the suction side and the pressure side are not tangent to each other at the trailing edge. We will now show that the Kutta condition entails another unexpected contradiction when the relative motion about the wing profile is unsteady.

Let P denote the wing profile, M a point describing P, $\vec{n}_M$ the unit vector normal to P at M in the inward direction, and P_e the profile described by the point M_e defined by $\overrightarrow{MM_e} = - \vec{n}_M(0+)$. Let B, B_e be the trailing edges of P and P_e, respectively (Figure 20).

Let (Ox, Oy) be a system S of axes moving with P, and let (O'x, O'y) be a system S' of fixed axes. We select S' so that it coincides with S at the time t under consideration. Let M' be the point of S' with which a point M of S coincides at t.

We suppose that the velocity $\vec{V}_E$ of O has on the X'-axis a component in the negative x'-direction and that the fluid is at rest at infinity ahead of profile P. Let $\vec{V}(M', t) = \nabla\Phi(M', t)$ be the velocity of the fluid point located at M at time t.

We have

$$\vec{V}(M', t) = \vec{V}_E(M, t) + \vec{V}_R(M, t) \text{ with } \vec{V}_E(M, t) = \vec{V}_E(0, t) + \vec{\Omega}_E(t) \wedge \vec{OM}$$

where $\vec{V}_R$ is the relative velocity of the fluid.

Let $\Gamma(t)$ be the circulation of $\vec{V}$ in a closed circuit surrounding the profile. If $\frac{d\Gamma}{dt} \neq 0$, a free vortex of intensity $- \frac{d\Gamma}{dt}$ dt is shed from B_e during the time interval [t, t + dt). This free vortex is carried by the fluid, and a fluid point $P(B_e(\tau))$ which was at B_e at $\tau < t$ is at t in a point $M_{f\tau}^t$ such that

$$\overrightarrow{B_e M_{f\tau}^t} = \int_\tau^t \vec{V}_R(P(B_e, \tau), t')\, dt'$$

Let t_0 denote the time at the beginning of the motion. The line L described by the points $M_{f\tau}^t$ since time t_0 makes a "cut" in the plane (Ox, Oy) in the sense of the theory of functions of a complex variable (x + iy).

Let L_1, L_2 be the lower and upper edges of the cut, respectively. Let μ_0 be a geometric point on L, and let $\vec{\nu}_{\mu_0}$ be the unit vector normal to L at μ_0 in the direction from L_2 toward L_1. Let μ_1, μ_2 denote the intersections of $\vec{\nu}_{\mu_0}$ with L_1 and L_2, respectively.

Let ℓ denote a path from μ_1 to μ_2 following first L_1 from μ_1 to M^t_{ft0} and then L_2 from $M^t_{ft_0}$ to μ_2; t_0 is the time of the beginning of the motion. Let P_0 be the fluid point located at μ_0 at t. This fluid point was at B_e at a certain time τ and

$$\int_{\ell} \vec{V}\,\vec{d}s = \int_{t_0}^{\tau} -\frac{d\Gamma}{dt}\,dt = -\Gamma(\tau) \tag{7.1}$$

Let Φ denote the velocity potential in the domain exterior to the closed contour $B_{e_1}M^t_{ft_0}B_{e_2}M_eB_{e_1}$. The points B_{e_1} and B_{e_2} are infinitely close to B_e on the pressure and suction sides, respectively, and the arc $B_{e_2}M_eB_{e_1}$ is the profile S_e itself. The velocity potential Φ is uniform in that domain since

$$\int_{B_{e_1}M_eB_{e_2}} \vec{V}\,\vec{d}s = \Gamma(t) = \int_{B_{e_1}M^t_{ft_0}B_2} \vec{V}\,\vec{d}s \tag{7.2}$$

but it is discontinuous across L since

$$\Phi(\mu_2', t) - \Phi(\mu', t) = \int_{\ell} \vec{V}\,\vec{d}s = -\Gamma(\tau)$$

Let P_0, P_1, P_2 be the fluid points respectively located at t at μ_0, μ_1', μ_2'. The trajectory of P_0 from τ to t is the arc $B_e\mu_0$ of L, and the velocities of the geometric points μ_1 and μ_2 are equal to the velocity of P_0. Let $\vec{V}(\mu_1, t)$, $\vec{V}(\mu_2, t)$ be the velocities of P_1 and P_2 at t. We have

$$\left.\begin{aligned}\left(\frac{\partial\Phi}{\partial t}\right)_{\mu_1', t} &= \left(\frac{d\Phi}{dt}\right)_{\mu_1', t} - \vec{V}(\mu_0, t) \cdot \vec{V}(\mu_1', t) \\ \left(\frac{\partial\Phi}{\partial t}\right)_{\mu_2', t} &= \left(\frac{d\Phi}{dt}\right)_{\mu_2', t} - \vec{V}(\mu_0, t) \cdot \vec{V}(\mu_2', t)\end{aligned}\right\} \tag{7.3}$$

Because of (7.2),

$$\frac{d\Phi}{dt}(\mu_2', t) - \frac{d\Phi}{dt}(\mu_1', t) = 0 \tag{7.4}$$

and (7.3) gives:

$$\begin{aligned}\frac{\partial\Phi}{\partial t}(\mu_1', t) - \frac{\partial\Phi}{\partial t}(\mu_2', t) &= -\vec{V}(\mu_0, t) \cdot [\vec{V}(\mu_1', t) - \vec{V}(\mu_2', t)] \\ &= -\frac{1}{2}[V_2^2(\mu_1', t) - V^2(\mu_2', t)]\end{aligned} \tag{7.5}$$

On the other hand, inside the domain occupied by the fluid, we have

$$\frac{1}{\rho} \nabla p_d(M, t) = -\frac{d\vec{V}}{dt}(M, t) = \left[-\frac{\partial\vec{V}}{\partial t} - \frac{1}{2}\nabla V^2 - \vec{\omega} \wedge \vec{V}\right]_{M, t}$$

There are paths from infinity ahead of the profile to μ_1' and μ_2' along which $\vec{\omega} = 0$. Consequently,

$$\begin{aligned}\frac{1}{\rho}[p_d(\mu_1', t) - p_d(\mu_2', t)] = -\left[\left(\frac{\partial\Phi}{\partial t}\right)_{\mu_1', t} - \left(\frac{\partial\Phi}{\partial t}\right)_{\mu_2', t}\right] - \frac{1}{2}[V^2(\mu_1', t) \\ - V^2(\mu_2', t)]\end{aligned}$$

From (7.5), we obtain:

$$\frac{1}{\rho}[p_d(\mu_1, t) - p_d(\mu_2, t)] \equiv 0 \tag{7.7}$$

This shows that <u>at every t, the pressure is continuous through L</u>. This result is in agreement with the general theory since the vortices located on L are <u>free</u>.

Let us consider now two points M_{e_1} and M_{e_2} respectively located on the pressure and suction sides of P_e. For any point M_e, we have

$$\frac{1}{\rho} p_d(M_e, t) = -\frac{\partial\Phi}{\partial t}(M'_e, t) - \frac{1}{2} V^2(M_e, t) + C(t)$$

$$= -\frac{d\Phi}{dt}(M'_e, t) + \vec{V}_E(M, t) \cdot \vec{V}(M_e, t) - \frac{1}{2}[\vec{V}_E(M, t) + \vec{V}_R(M_e, t)]^2 + C(t)$$

$$= -\frac{d\Phi}{dt}(M'_e, t) + \frac{1}{2} V_E^2(M, t) - \frac{1}{2} V_R^2(M_e, t) + C(t)$$

Consequently,

$$\frac{1}{\rho}[p_d(M_{e_1}, t) - p_d(M_{e_2}, t)] = -\left[\frac{d\Phi}{dt}(M'_{e_1}, t) - \frac{d\Phi}{dt}(M'_{e_2}, t)\right]$$
$$+ \frac{1}{2}[V_E^2(\mu_1, t) - V_E^2(\mu_2, t)] - \frac{1}{2}[V_R^2(M_{e_1}, t) - V_R^2(M_{e_2}, t)] \qquad (7.8)$$

Thus when $M_{e_1} \to B_{e_1}$ and $M_{e_2} \to B_{e_2}$, we obtain

$$\frac{1}{\rho}[p_d(B_{e_1}, t) - p_d(B_{e_2}, t)] = -\frac{d\Gamma}{dt} - \frac{1}{2}[V_R^2(B_{e_1}, t) - V_R^2(B_{e_2}, t)] \qquad (7.9)$$

But because of the Kutta condition, we have

$$|V_R(B_{e_1}, t)| = |V_R(B_{e_2}, t)| \qquad (7.10)$$

The common value of these two relative velocities is zero if both sides of P_e are not tangent to each other at B_e.

Consequently

$$\frac{1}{\rho}[p_d(B_{e_1}, t) - p_d(B_{e_2}, t)] = -\frac{d\Gamma}{dt} \qquad (7.11)$$

It can easily be shown that μ_{e_1} on L_1 and μ_{e_2} on L_2 are infinitely close to B_{e_1} and to B_{e_2}, respectively,

$$\left.\begin{aligned} \frac{1}{\rho}\left[p_d(B_{e_1}, t) - p_d(\mu_{e_1}, t)\right] &= -\frac{1}{2}\frac{d\Gamma}{dt} \\ \frac{1}{\rho}\left[p_d(B_{e_2}, t) - p_d(\mu_{e_2}, t)\right] &= +\frac{1}{2}\frac{d\Gamma}{dt} \end{aligned}\right\} \qquad (7.12)$$

I am unaware as to whether the rather surprising results expressed by Equations (7.11) and (7.12) have previously been mentioned by other investigators. The jump of the pressure from the pressure side to the suction side at the trailing edge of a wing profile in unsteady relative motion is an effect of the apparent added masses due to the shedding of a free vortex sheet. At the instant t_1 of the beginning of a finite perturbation, this jump is infinite for $\left(\frac{d\Gamma}{dt}\right)_{t=tz+0} = 0\left(\frac{1}{\sqrt{t-t_1}}\right)$. It probably decreases with the aspect ratio. We have shown [8] that in the case of wing profiles, the apparent added masses considerably affect the pressure distribution on the profiles. We may wonder whether it is not a source of vibration and noise in the case of wings or of propeller blades with large aspect ratio.

SNAPROLL

When a submarine is in a turning motion about a vertical axis, the system of its inertial forces and of the hydrodynamic forces exerted on it contains a couple whose axis is longitudinal. This couple is due to the centrifugal force which is horizontal and passes through the center of gravity G and to a centripetal force which is equal and opposite in direction to the former and passes through the center of volume C of the submarine. As G is located below C, this couple creates a heeling moment to the interior side. Since the motion of gyration is delayed with respect to the time at which the rudder has reached the ordered angle, one would expect the heel to increase monotonically until the existing couple is cancelled by the couple due to the weight and to the hydrostatic force.

However, this is not what happens with modern high-speed submarines. In general, these ships are fitted with two rudders approximately symmetric with each other in terms of a horizontal plane and with two diving planes symmetric in terms of the longitudinal plane. The vessel heels (to port if

the rudders go to port) almost immediately after the rudders reach the desired angle, and a strong overshoot occurs. The maximum of the roll angle may be twice or three times the final heel angle, which is reached after some roll oscillations. Figure 21 roughly sketches the phenomenon.

Of course, when the rudders are at the angle α, the incident flow on the diving planes is no longer symmetrical with respect to the longitudinal plane of symmetry of the vessel. Thus we can ask whether the rudders may generate a sheltering effect on the diving planes, consisting of a lift in the upward direction on the starboard diving plane and in the downward direction on the port diving plane. A simple experiment on an isolated set of two rudders and two diving planes could probably determine whether the above explanation is correct. Personally, I feel that another explanation is also possible. The horizontal lift of the rudders is generated quasi-instantaneously and it varies like V_E^2. In the first phase of the motion, the vessel experiences an acceleration to starboard (if the rudders are to port) and after a time τ very close to the beginning of the motion, the sail has an angle of attack proportional to $V_E\tau$ (Figure 22). The quasi-steady lift on the sail is thus $O(V_E^3\tau)$. The real lift on the sail is greater because of the effect of the apparent added masses due to the free vortices shed by the sail as mentioned in the previous section. Possibly this may generate a heeling moment of importance and thus causes the overshoot shown by experiments.

"PITCH UP" OF SUBMARINES

When a submarine is turning in a horizontal plane, the system of hydrodynamic forces exerted on it contains a moment about a transverse axis. The effect of this moment seems to be an increase in the apparent weight of the stern or a decrease in that of the bow.

In the past, the asymmetry of the submarine with respect to the horizontal plane of the center of volume was important. There was one rudder only, and the pitch-up condition could be explained by a suction on the flat part of the hull above the rudder due to the circulation around the rudder.[9]

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[9] Isabelle, M., "Remarques sur une cause perturbatrice de l'équilibre dynamique d'un sous-marin en plongée," Bull. Ass. Tech. Mar. et Aéron., Paris (1938).

Today's high-speed submarines are much more symmetrical and the interactions with the hull of each of the two rudders, which are in opposite directions, could cancel each other. However, a pitch-up moment still exists. In fact, the heel caused by the gyration modifies the structure of the bound vortex half-rings on the forebody since the bow is inside the circle of gyration. The suction side on the forebody is starboard if the gyration is to port and because of the heel a lift in the upward direction may appear on the forebody.

COURSE STABILITY

The course stability of the motions in a horizontal plane and also in the vertical plane are very important properties of a ship. The phenomenon of instability of certain surface ships has been known for many years, but it would be impossible to understand the phenomenon without the help of an approximate theory. This theory has suggested means for experimentally determining the forces acting on the hull, and now the condition to be satisfied is rather well known. Although the control systems now available permit the ship to keep a route closer to the required one, it is still desirable that ships be naturally stable. Stability decreases the power necessary to sustain speed. In the case of submarines, the instability in the vertical plane is dangerous and the instability in the horizontal plane induces sway and yaw which generate rolling motions.

The forces exerted on a ship model are generally measured in conditions which do not permit the history of the motion to be taken into consideration. Thus stability is studied by assuming that the forces are quasi-steady. The equations are differential rather than integral-differential as they should be.

In his thesis, which concerns surface ships only, Casal attempted to take into account the terms which depend on the history of the motion. He concluded that the effects of these terms are rather small; in general, the stability - or the instability - would be reinforced if, according to the differential equations, the ship is stable or unstable. Nevertheless, he showed that in certain circumstances, the integral-differential terms may decrease the damping coefficients so that the ship has self-yawing and swaying motions.

The Casal conclusions are based on the concept that during small motions, all the free vortex filaments are gathered in a unique free vortex filament shed from the stern post. For reasons explained in Chapter 6, it seems to me that this scheme is too simple and that the effect of the integral-differential terms needs new refinements.

EXPERIMENTAL STUDIES ON MANEUVERABILITY

All that we know on the subject of maneuverability comes from experience acquired either by observing ship motions in a seaway or by carrying out tests on ship models.

At the beginning, the main problem was to determine the characteristics of the steady gyration of surface ships. The hydrodynamic forces exerted on the models were not measured. Then course stability was considered. Recall the mention in Chapter 1 that criteria were formulated by Contensou[2] and by Davidson and Schiff[3]. Unfortunately these criteria were difficult to apply because they were expressed in terms of the forces exerted on the ship and such forces were unknown. A finding by Dieudonné[4] is very important, namely, that a ship is stable only if the resulting force in a motion of translation at a very small angle of attack pierces the longitudinal plane of symmetry at a point located downstream from the intersection with that plane of the resultant of the centrifugal force and of the hydrodynamic transversal force in a gyration at zero drift angle. Dieudonné also showed that this condition means that the ratio $\omega L/U$ in steady gyrations is a uniform function of the rudder angle δ. It has been mentioned in the previous section that this condition does not take into account the history of the motion and that in some "pathological" cases, self-yawing motions may happen at zero rudder angle. Very simple tests in a rectilinear tank or (and) in a steering tank will suffice to determine whether a ship model is stable or not.

The planar motion mechanism originated at DTNSRDC more than 15 years ago was designed to measure the hydrodynamic forces exerted on submarine models either in oblique translation in the horizontal plane and in the vertical one or in forced periodic swaying, yawing, heaving, and pitching motions. The torque about the longitudinal axis of the model can also be measured simultaneously. The planar motion mechanisms permit determining the response of a free-running model to slight variations of the rudder or of the diving

planes under the assumption that the hydrodynamic system of forces reduces to the quasi-steady one. Brard has shown[10] that this technique can also provide informations on the effect of the history of small motions about a uniform motion of translation provided that the range of the ω values is sufficiently large. If the reduced angular velocity $\omega L/U$ acts on the dimensionless system of forces, the part due to the added masses has to be derived from the measurements at high $\omega L/U$ values and the out-of-phase forces from the measurements at small values of this parameter.

Planar motion mechanisms for surface ships are now available at several establishments; generally the amplitudes of the horizontal periodic motions are sensibly greater than those of the motions permitted by the initial variants of this device.

Tests on models in forced motions do not fill all the needs for several reasons. In the first place, the present state of the theory does not permit predicting all the phenomena to which the maneuvering ship can give rise. This is clearly the case when the problem is to determine the behavior of the ship in waves, in restricted waters, or during fierce breaking. Perhaps this is still more evident when six degrees of freedom have to be considered simultaneously, as, for example, in the case of submarines or of nonconventional ships such as hydrofoil craft or surface effect vehicles. A review[11] is available of the various methods used in the Paris Naval Tank together with a description of the free submarine models that we are testing on the bay of St. Tropez on the French Mediterranean coast for experimental determination of the maneuvering qualities of submarines.

The purpose of such tests of free-running submarine models is not only to compare the real motions to those computed by using analog or (and) hybrid computers; we also wish to discover all the unexpected phenomena which may occur, for example, in maneuvers to be attempted in cases of emergency.

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[10] Brard, R., "A Vortex Theory for the Manoeuvering Ship", Proc. Fifth Symposium on Naval Hydrodynamics, Bergen (1964).

[11] Brard, R., et al., "Le modèle libre de sous-marin du Bassin d'Essais des Carènes," Bull. Ass. Tech. Mar. et Aéron., Paris (1968).

FINAL REMARKS AND CONCLUSION

During the last two or three decades considerable progresses have been made in the field of ship maneuverability. They have resulted, essentially, from the pressure of needs. Today the final purpose is very ambitious since it is to control any maneuver in a complete manner. Desired changes of course (or of depth in the case of submarines) have to be performed according to laws programmed in advance. For instance, the time allowed for reaching the new heading, the maximum angular velocity of the rudder, the magnitude of the possible overshoot, are assigned, so that the ship must maneuver as she were on a rigid trajectory. This implies that various types of trajectories have been completed, that one of them is selected on the occasion of each maneuver and that the control system is completely automated.

For designing such a control system and computing the roads of the ship, it is necessary to know how the ship responds to any given maneuver. This can be done by integrating the equations of the motion, which implies that the hydrodynamic forces exerted on the hull and on the moving control surfaces have been determined in a number of conditions.

In the last Section of Chapter 7 mention has been made of several experimental methods which can be used to that end. Measurements of forces on captive models in forced motions give an important part of the necessary information. But the estimates so obtained obviously differ from the forces which would be exerted on free-running models. Furthermore, certain forces are not attainable experimentally. Resort to theory is useful for obtaining at least the order of magnitude of terms missing in the equations. Theory is also of interest for yielding the structure of the equations. In case of discrepancy between measured and calculated forces or between observed and computed roads, one might be tempted to include into the expression of forces terms the theoretical examination of which would reveal that they are physically meaningless. There is a risk that the equations after modification lead to worse results in some other conditions.

Vortex theory is the only one which explains how force orthogonal to the direction of the motion can be created in an almost inviscid liquid. It therefore seemed to me that vortex theory should be the primary tool for

building up a mathematical maneuverability theory. This is the problem investigated in the present lectures.

Before attempting to extend the vortex theory to ships, I have believed it important to show that it permits to evaluate the system of forces due to the inertia of the liquid, that is the forces due to the so-called added masses. The classical theory of added masses is briefly recalled in chapter 2, while chapter 3 treates in detail the corresponding vortex theory from both kinematic and dynamic points of view. Attention is to be paid to the splitting up of the vortex distribution into two parts. One induces zero velocity field outside the body, whereas the second generates outside the velocity potential due to its motion.

Chapter 4 extends the wing theory to wings with a finite thickness and small aspect ratio and to bodies in a motion of translation. In the case of a relatively thick wing the Kutta condition is interpreted as due to the shock between streamlines coming from the suction side and those coming from the pressure side. When the flow is steady, the relative velocities on both sides are equal and opposite at the trailing edge. This paradoxical result is due to the fact that the thickness of the boundary layer is infinitely small in an almost inviscid liquid.

The problems relating to ships in a uniform, steady, oblique motion of translation give rise to an analysis more intricate than that of wings because there generally exists on the hull of a ship no line which can be considered as the trailing edge of a wing, at least from a geometrical standpoint. However, in the case of a submarine moving in its vertical plane of symmetry, the visualization of the flow seems to show that the relative streamlines passing on the upper side and those passing on the bottom intersect along a U-shaped line open in the direction of the motion. This turns out to generate shocks and thus the shedding of a free vortex sheet. This phenomenon is analogous to the one just been mentioned in the case of thick wings (but, in the present case, for calculating the flow around the ship, the place on the hull of its "trailing" edge should be determined after visualizing the flow).

I hope that the reader will easily accept my explanation of the existence of a vortex sheet downstream from the body. As in the case of a wing, the

problem would remain unsolved if no additional condition were prescribed. This is what we call the generalized Kutta condition. It states that the pressure on the hull is continuous through the shedding line. It takes the form of an integral-differential equation similar to that encountered in the case of wings with a finite aspect ratio, but it is sensibly more complicated and it would probably be very difficult to solve mathematically. In practice this should not be of very great importance since all the necessary information could be derived from measurements and observations carried out on models in uniform translation (provided that several angles of heel are used).

In the case of oblique translation in the horizontal plane, the streamlines on the port and starboard sides also seem to intersect so that free vortices are shed from the rear part of the keel and from the stern post. The same scheme also applies to submarines moving in a horizontal plane. A generalized Kutta condition determines the problem completely. Although apparently simpler than the condition for motions in the vertical plane, it would probably lead to many computational difficulties. But pure forced swaying motions should also yield all the information needed for practical purposes (provided that the tests include several values of the trim and heel angles).

Chapter 5 was initially planned to deal with surface ships. The arcs of vortex filaments, the union of which forms the vortex sheet located on the hull, pearce the free surface. To determine the velocity induced by the vorticity a vector potential is needed. If the free-surface condition is linearized, it is possible to continue the real vortex field above the free surface by taking its mirror image with respect to the plane of the free-surface at rest. In his thesis, Casal had reduced the part played by the free surface to that of mirror. Furthermore, he had assumed that the ship behaved as an infinitely thin flat wing, the bound vortex filaments being vertical segments of straight lines in the centerplane. By means of a complementary assumption concerning the directions of the free vortices near the hull, he obtained an approximate solution with many interesting features. But the boundary condition at the hull was satisfied only along the intersection of the centerplane with the plane of the free surface at rest.

Casal's scheme therefore implied that the draft was very small. We have shown here that it is possible to satisfy the boundary condition on the whole surface of the centerplane, but this requires a Kutta condition. Because of the smallness of the beam/length ratio, the procedure sketched at the end of Chapter 4 for double models is no longer usable. Finally, we have arrived to the conclusion that the approximation of infinitely thin hulls is much too crude. For this reason, we have come back to normal ship forms in the second part of Chapter 5. It has been shown that the assumption that the free surface condition can be linearized leads to a solution involving a Kelvin singularity distribution on the contour of the waterline.

Chapter 6 tackled the problem of small motions first in a nearly vertical plane and then in a nearly horizontal plane, i.e., the so-called motions of the first and the second kinds, respectively. Of course, the generalized Kutta condition gives for $d\Gamma/dt$ along the shedding line an integral equation of mixed Volterra and Fredholm types. Despite the more complicated structure of the equations, small motions of the first kind are on the whole similar to small motions about a wing with a finite aspect ratio. The vortex wake consists of a vortex sheet.

For an analogous phenomenon to happen in the case of small motions of the second kind, the free vortices should be shed from the sternpost alone. But it is not so, in general. For instance, in a pure swaying motion, they are shed from a narrow region close to the keel line, alternately to port or to starboard, according as the component u_2 on the transverse axis of the velocity of the center of gravity is in the negative or in the positive direction. Consequently the vortex wake does no longer consist of a surface, but of two sequences of domains, each of finite volume, one created to port and the other to starboard. In a pure yawing motion u_2 is zero at any time, but the angular velocity u_6 is alternately positive and negative. When it is positive, the incident velocity comes from the starboard side at the forebody and from the port side at the afterbody. During each time interval corresponding to $u_6 > 0$, two vortex wakes of finite volume are shed to starboard from the forebody and to port from the afterbody. When $u_6 < 0$, the volumic wake shed from the forebody is to port and that shed from the afterbody is to starboard.

Chapter 7 deals with the interaction between hull and appendages, particularly those playing an important part in maneuver: rudder, diving planes, fixed fins. Vortex theory explains why certain fittings may increase their efficiency and make the response of the ship quicker. Tentative explanations have also been proposed as for unexpected phenomena (snap-roll, pitch up of submarines). The last Section presents a brief survey of experimental methods used in the field of ship maneuverability.

Let us come back now to the fundamentals of maneuverability theory considered as an application of the vortex theory.

In two-dimensional flows around wing profiles, the key to the problem is yielded by the well-known Kutta-condition. One might justify it by putting forward that, if this condition were not satisfied, the velocity at the trailing edge would be infinite. However such a reason has not always seemed to be decisive. I remember to have heard Prandtl speaking about this question at the 3rd Congress of Applied Mechancis (Stockholm, 1931). What has entailed the firm belief of the attendents, has been a moving picture showing how a vortex appears at the trailing edge, how it grows and then travels with the flow, followed by a vortex wake which progressively vanishes far downstream from the profile when the circulation around the latter becomes a constant. If the fluid were inviscid, no vortex could raise, except in case of shocks (Helmholtz theorem). In a real fluid, however, particles coming from the pressure side tend to wind around the trailing edge instead of flowing back along the suction side in the direction of the rear stagnation point. So a shock happens. The phenomenon is connected with viscosity effects. Lagrange's theorem according to which the circulation in any closed fluid circuit is a constant applies in the regions where the viscosity effects are negligibly small. So the circulation in a fluid circuit surrounding the profile and the vortex wake born at the trailing edge remains zero indefinitely.

In the case of a non infinitely thin wing with a finite aspect ratio, it has been seen in Chapter 4 that, because of Stokes' theorem, the velocities on the two sides of the wing, near the trailing edge, are in opposite directions. When the aspect ratio tends to infinity, both tend to zero, which gives again the Kutta condition. But before the limiting situation is reached, shocks have appeared at the trailing edge.

The assumption underlying the theoretical considerations developed in the present lectures is that the existence of a lifting force on a ship hull implies that of free vortex filaments shed from a line acting like the trailing edge of a lifting surface. But the theory does not yield the exact position of this line. The requirement to be met is the continuity of the pressure across the free vortex sheet, close to the hull. The equation expressing this condition depends on the hull geometry and also on the motion. There is no need to draw the attention to the formidable intricacy of the computational problem.

Nevertheless, vortex theory knows some successes of importance in the field of ship maneuverability. For instance the course stability means that the hull in forced gyration at zero drift angle experiences a lift which is not much smaller than the centrifugal force. Otherwise the resultant of the two could not pearce the centerplane ahead of the intersection with the latter of the lift in a translation at a very small drift angle. The problem related to course stability could not be tackled by the older methods recalled at the end of Chapter 2.

To conclude these lectures, I should like to say that my firm opinion is that neither pure empiricism alone, nor pure theory alone can provide us with means sufficient for resolving all the problems relevant to ship maneuverability. But I am convinced that by making a judicious use of experiments and of theoretical schemes, many points still rather obscure will be gradually removed.

ACKNOWLEDGMENTS

The writer again expresses his gratitude for the opportunity to deliver this Second Series of David W. Taylor Lectures. Particular and warmest thanks go to Captain P. W. Nelson, Commander, David W. Taylor Naval Ship Research and Development Center and to Dr. W. E. Cummins, Head of the Ship Performance Department.

Very often in the past I have visited the Center and had opportunities to discuss with Dr. Cummins and his staff various problems related to ship hydrodynamics. I enjoyed each of these visits. But until now I had not stayed here so long. This five week sojourn permitted me to take full advantage of the hospitality of the U. S. Navy.

My text benefited from the comments of Mr. Justin McCarthy and of Dr. Ming S. Chang both of the Hydrodynamics Branch; both, furthermore, greatly improved my English. I wish to emphasize how deeply their help was appreciated. I also express my warm thanks to Mrs. Barbara Raver who so carefully typed the first draft of my manuscript.

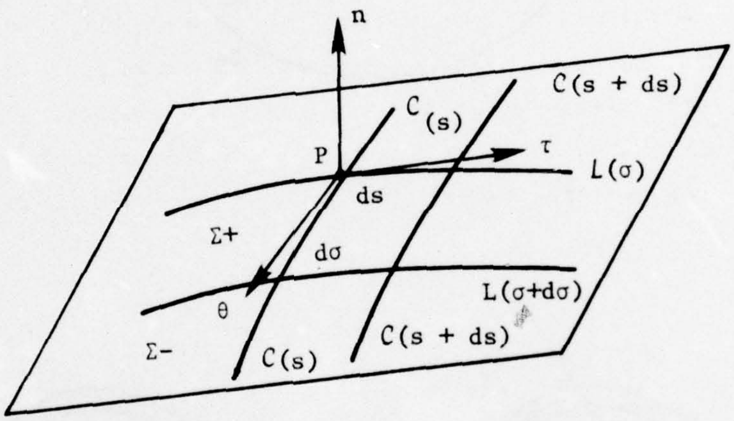
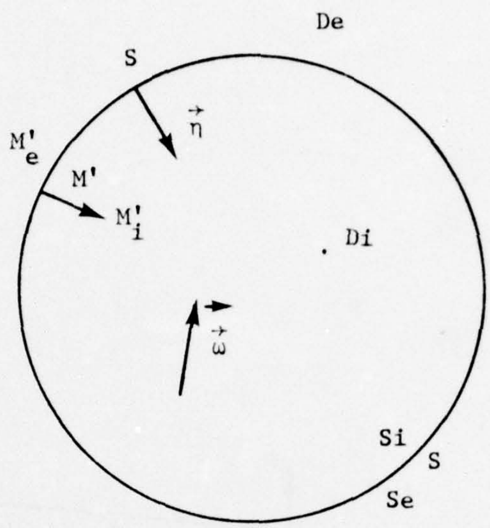


Figure 1-Local Coordinate System on a Vortex Sheet

Figure 2 - Geometry for Poincaré Formula



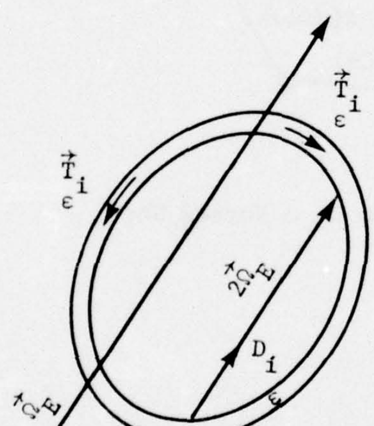


Figure 2 a - Geometry for Vortex Distribution $\mathcal{D}$ equivalent to a Moving Body

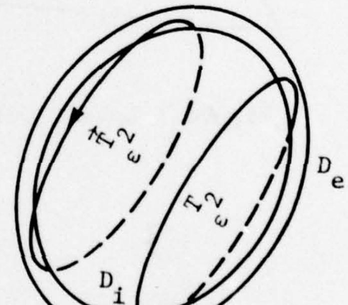


Figure 2 b - Vortex Distribution $\mathcal{D}_2$

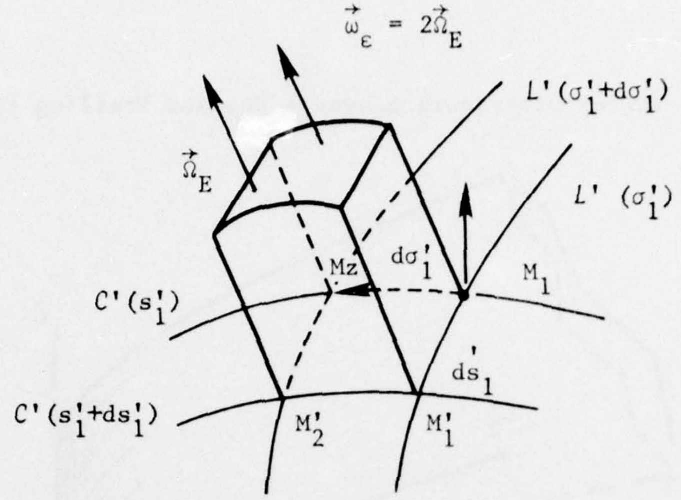


Figure 2 c -Part of Vortex Distribution $\mathcal{D}_1$ ($\mathcal{D}_1 = (\Sigma,\ T_1/\varepsilon) + (D_i,\ 2\vec{\Omega}E)$)

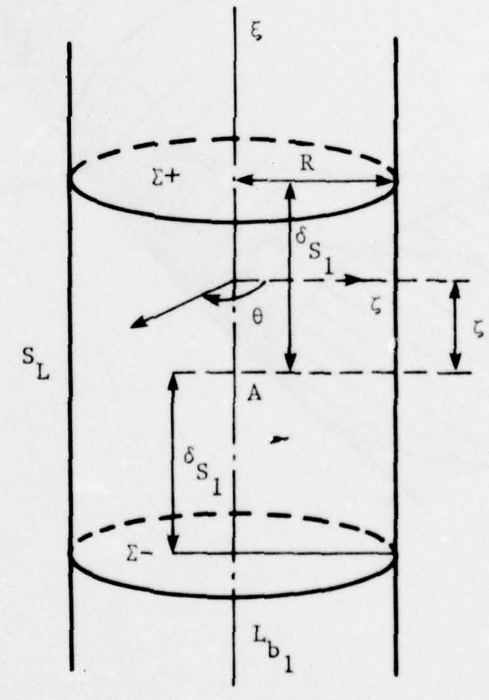
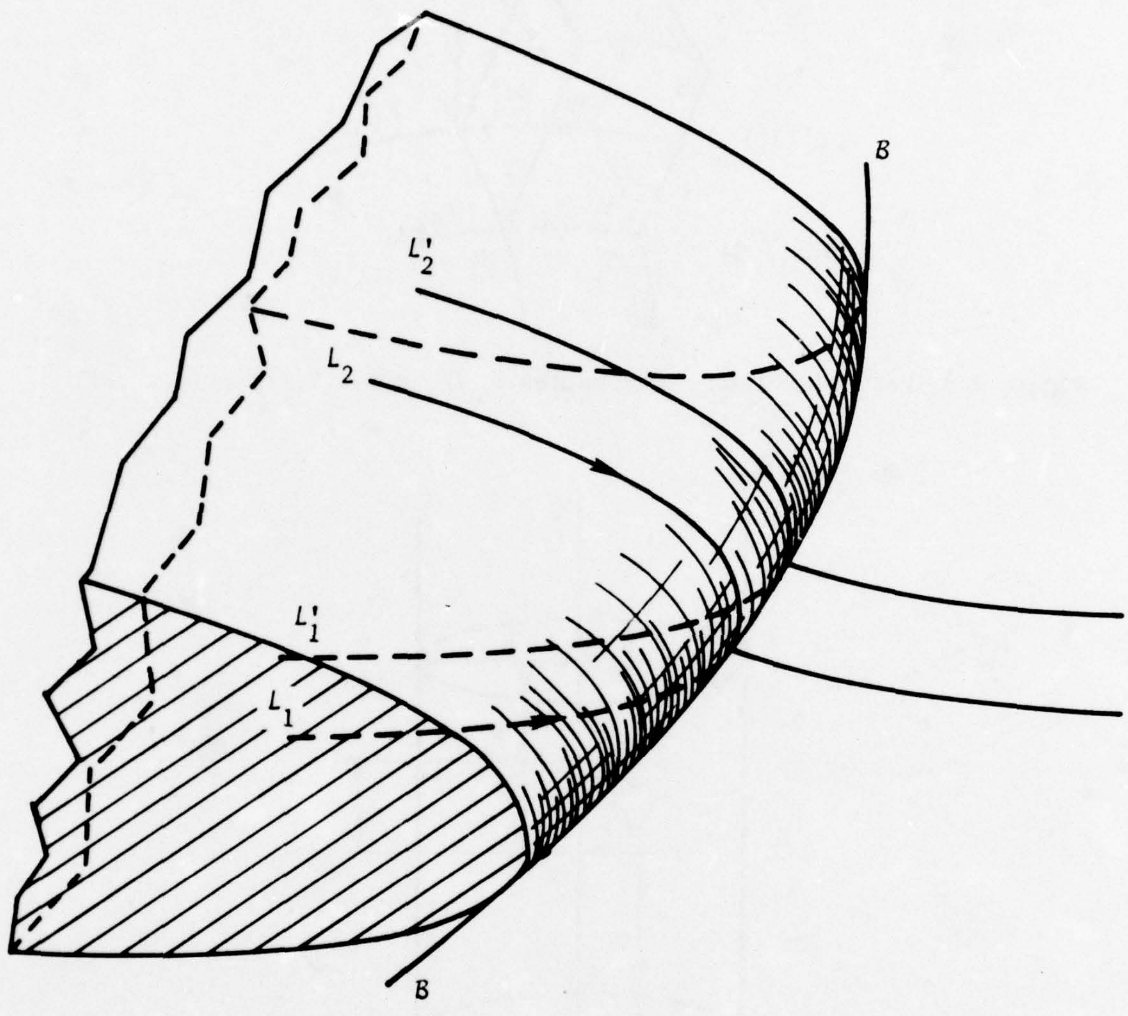


Figure 3 - Figure for Formulas (3.33)-(3.42)

Figure 4 - Vortex Distribution over a Rounded Trailing Edge of Wing



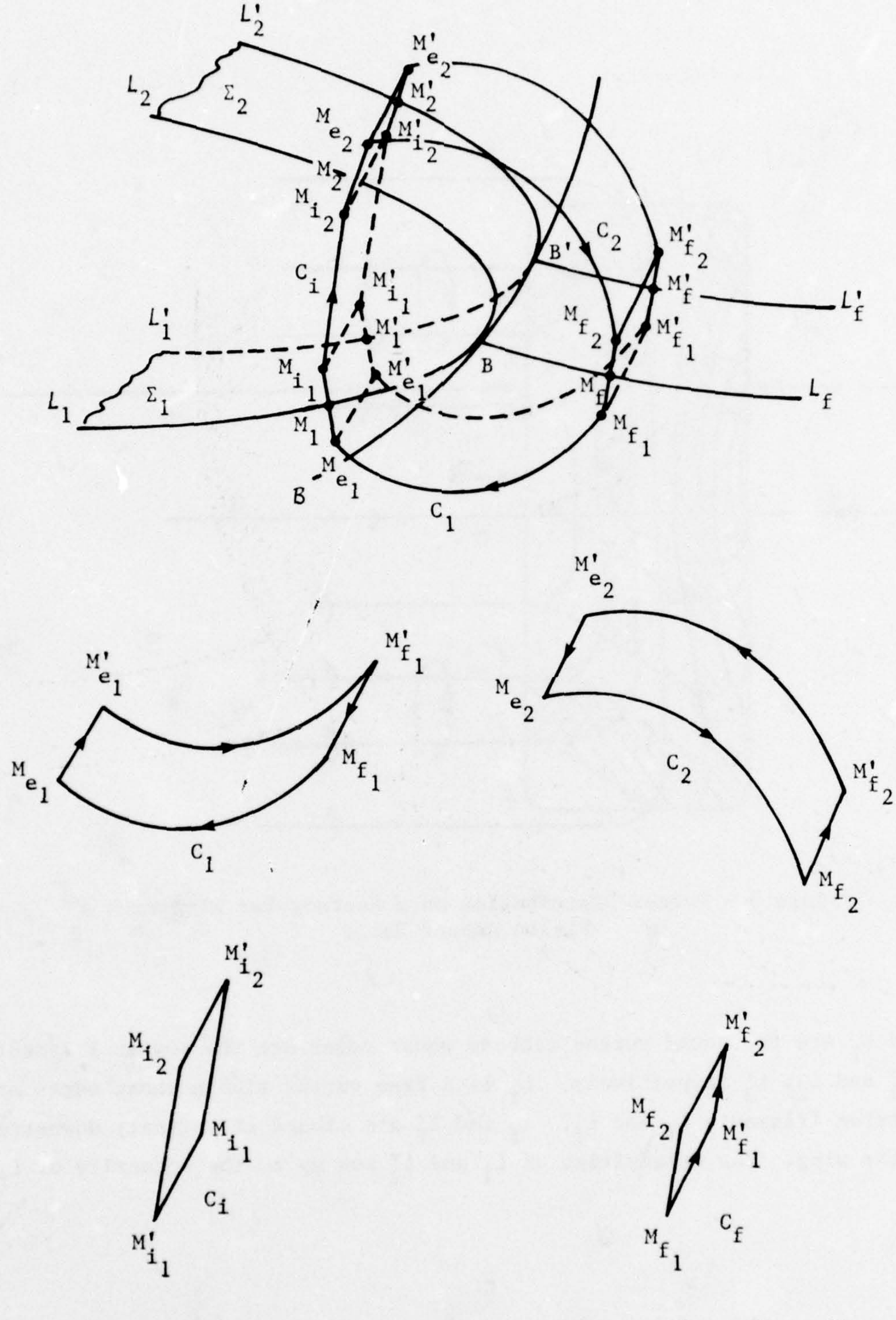


Figure 4 a - Figure for Establishing Formula $d\Gamma_f = d\Gamma_1 + d\Gamma_2$

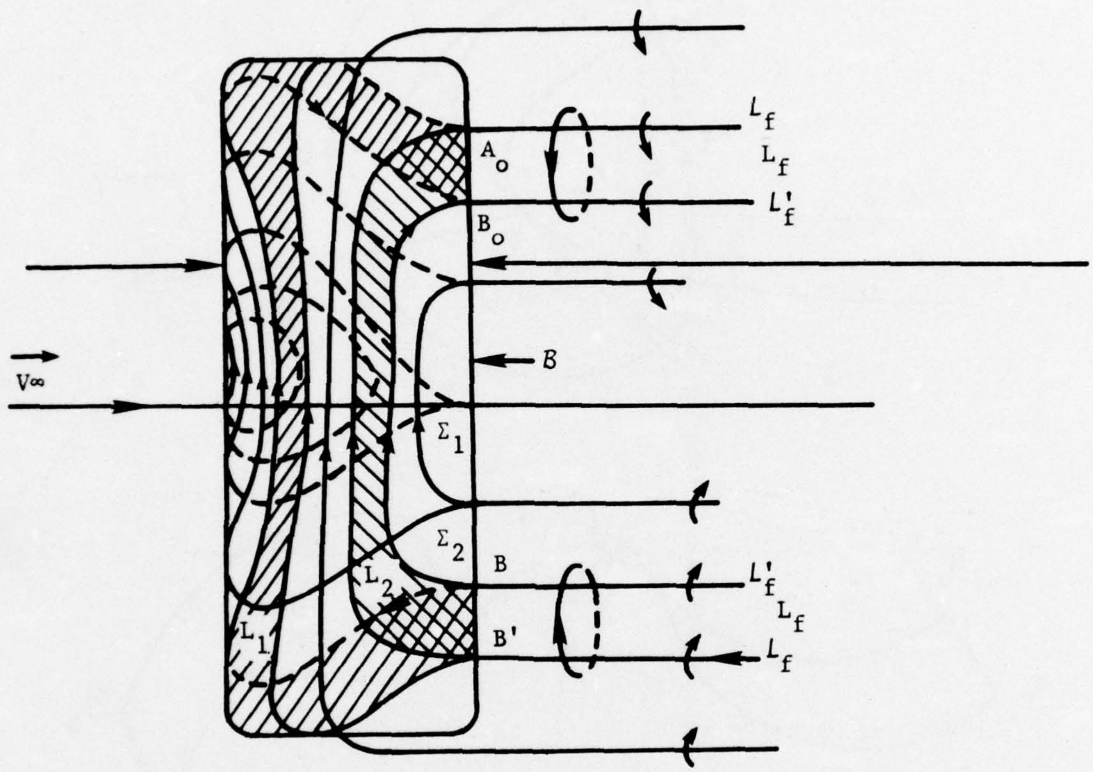


Figure 5 - Vortex Distribution on a Rectangular Wing with a Finite Aspect Ratio

L_1 and L_2 are two bound vortex ribbons whose edges are the vortex filaments L_1, L_1' and L_2, L_2' respectively. L_f is a free vortex ribbon whose edges are the vortex filaments L_f and L_f'. L_f and L_f' are closed at infinity downstream from the wing. The intensities of L_1 and L_2 sum up to the intensity of L_f.

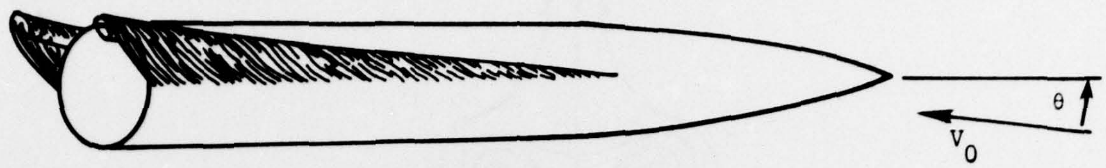


Figure 6 - Free Vortex Distribution for an Axisymmetric Body

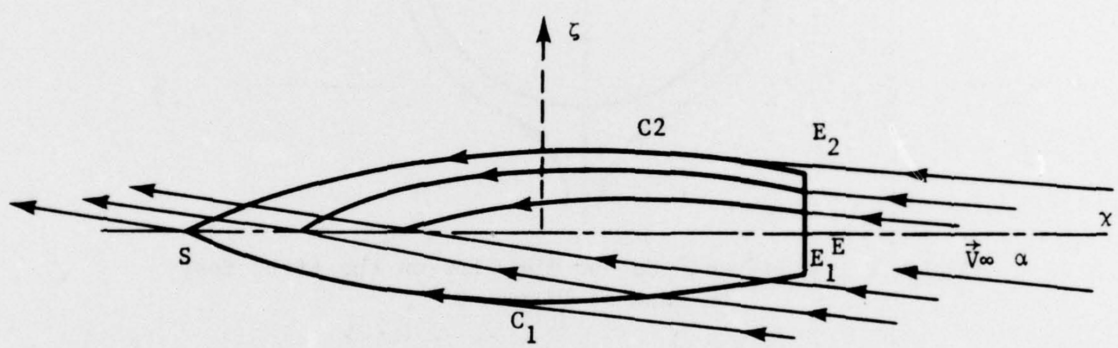


Figure 7 - Shocks between Upper and Lower Streamlines on the Hull of a Submarine
(Case of a small angle of attack)

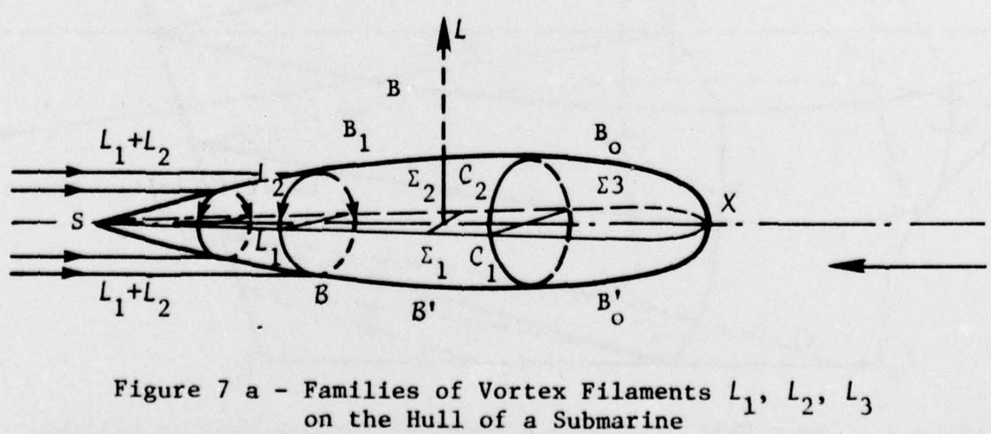


Figure 7 a - Families of Vortex Filaments L_1, L_2, L_3 on the Hull of a Submarine

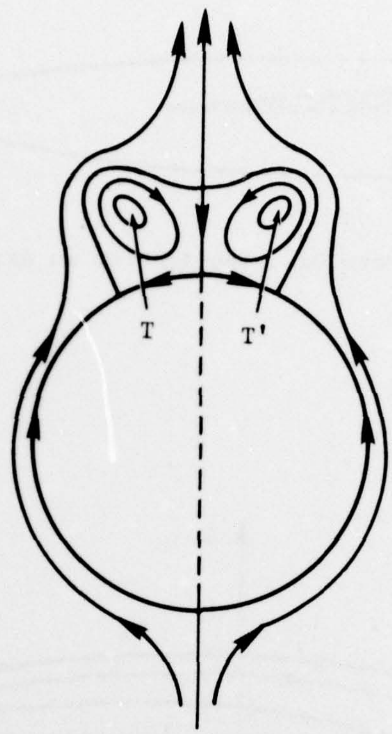


Figure 8 - Transverse Cut of the Flow on the After Body
in Case of Figure 6

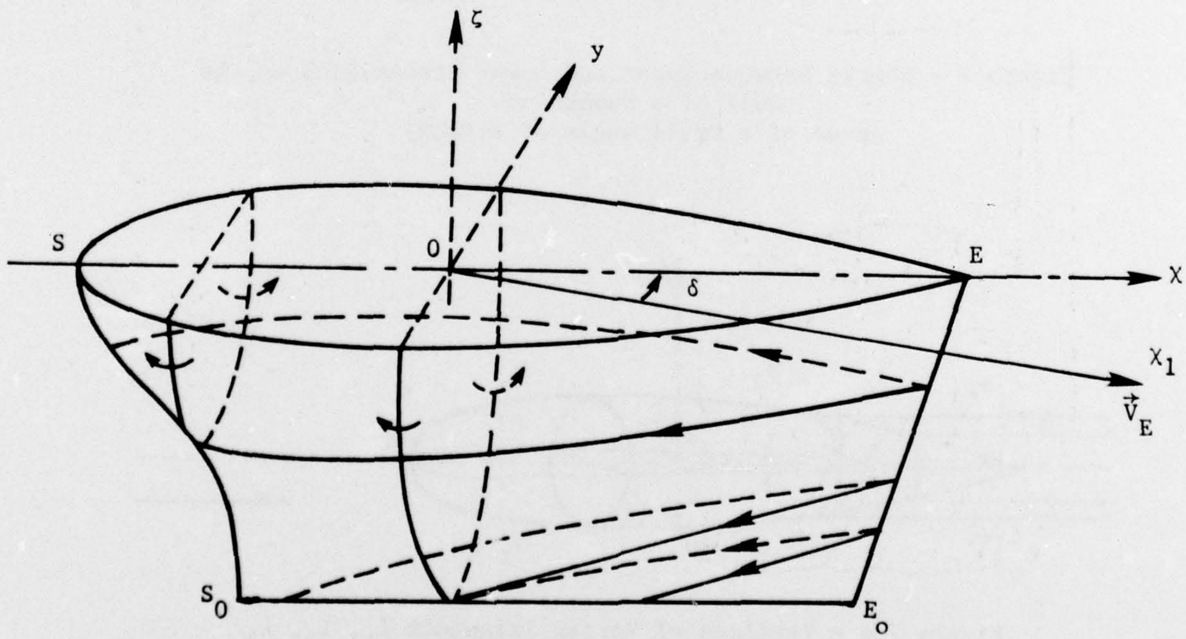


Figure 9 - Lower Half of a Double Model of Surface Ship in Oblique
Translation With a Positive Drift Angle δ
(Bound vortex filaments and relative streamlines on the hull)

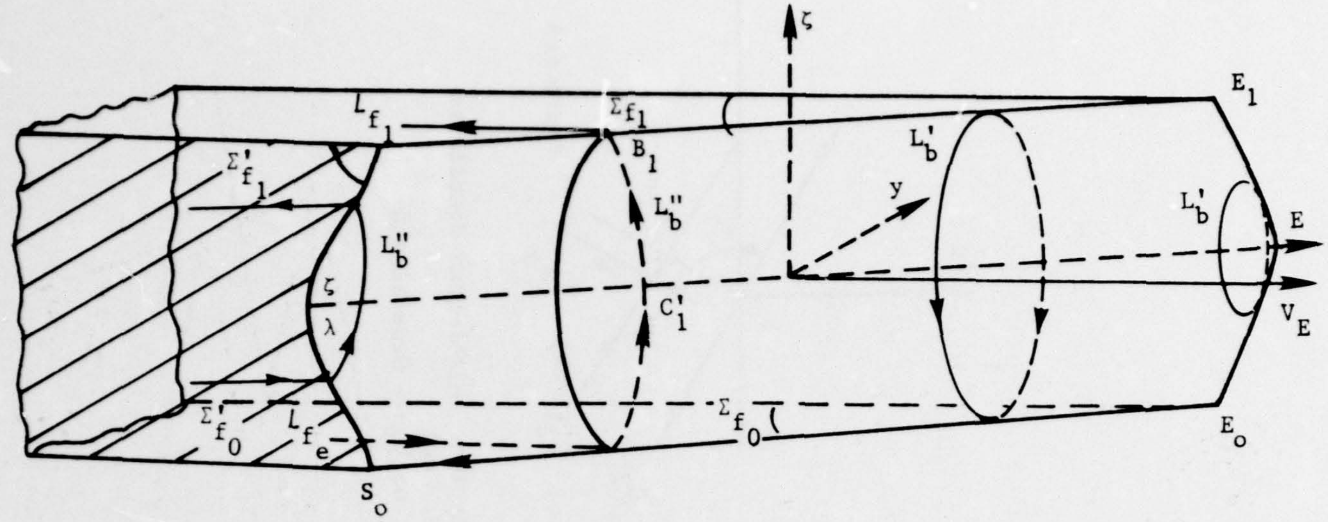


Figure 10 - Double Model of a Surface Ship in Oblique Translation

$\Sigma_f = \Sigma_{f_o} + \Sigma_{f_1}$ Free vortex sheets

Σ'_{f_o}, Σ'_{f_1} Parts of Σ_{f_o}, Σ_{f_1}, respectively, ending or beginning at sternpost.

Σ_o, (Σ_1), lower (upper) half of the hull Σ of the double model

Σ^+, (Σ^-), port (starboard) side of Σ

$L' = L'_b$ Vortex rings on $\Sigma^+ + \Sigma^-$

$L'' = L_{f_o} + L''_b + L_{f_1}$ U-shaped vortex filament L''_b lies on Σ^+.

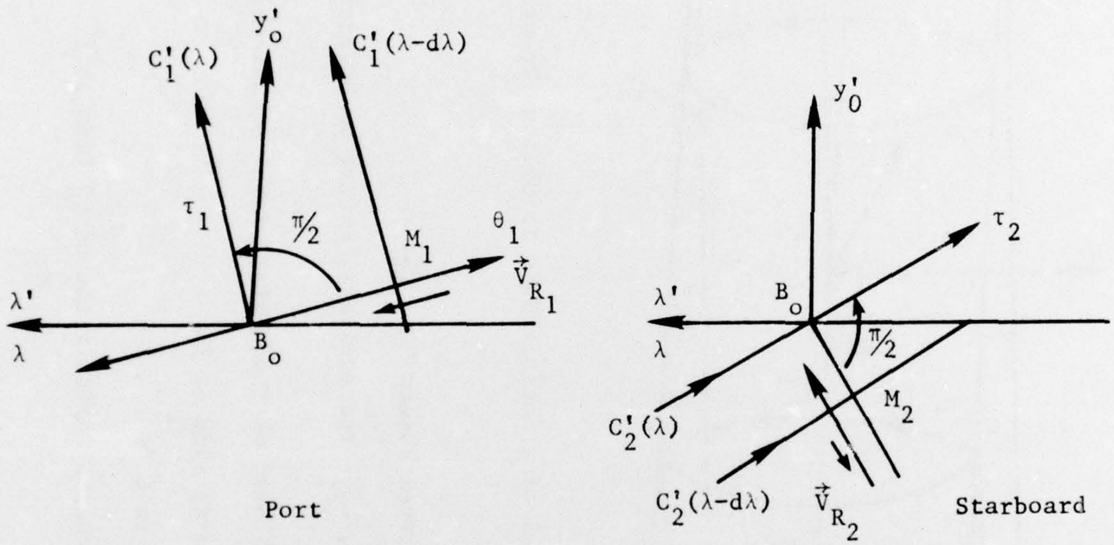


Figure 11 - Ship in Oblique Translation. Right-hand Coordinate System at the Keel
(Case of a rounded or flat bottom - Generalized Kutta condition)

$\mathcal{B}$: Shedding line on the hull

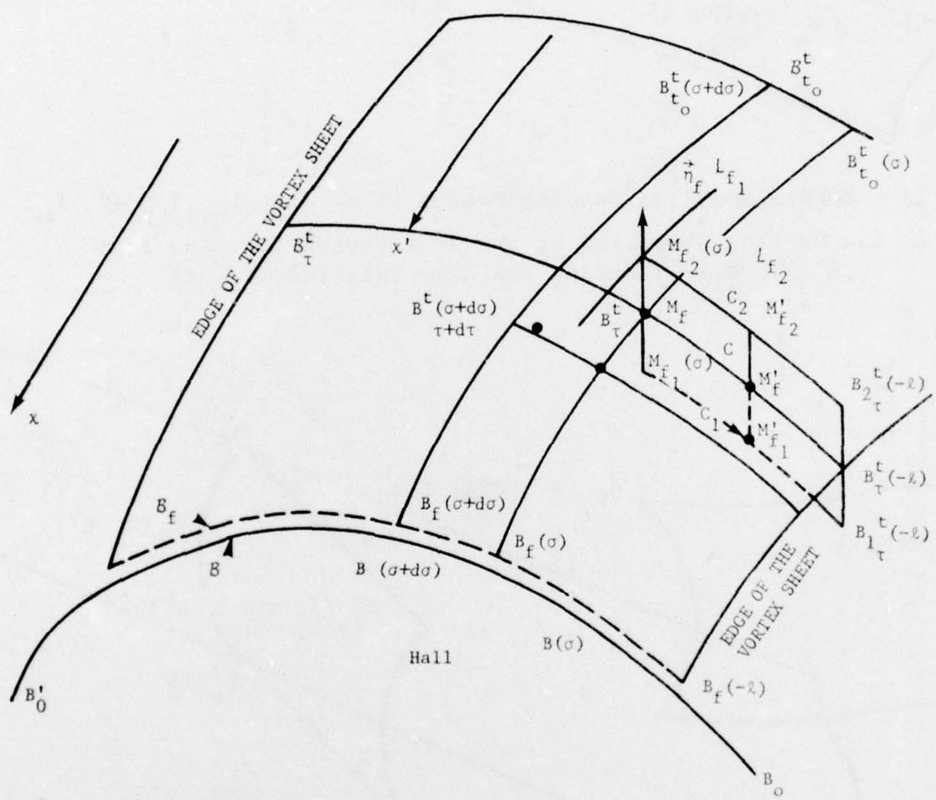
$\mathcal{B}_f$: Line on Σ_f infinitely close to $\mathcal{B}$

$\mathcal{B}^t_\tau$: Position at t of the fluid line which was on $\mathcal{B}_f$ at τ.

E, E': Lateral edges of Σ_f.

σ : Curvilinear abscissa on $\mathcal{B}$ and $\mathcal{B}_f$.

$B^t_\tau(\sigma)$: Position at t of the fluid point which was on $B_f(\sigma)$ at τ.



$B_f(\sigma)B^t_{i_o}(\sigma)$: Arc of Σ_f described by the geometric point $B^t_\tau(\sigma)$ when τ increases from t_o to t.

$B^\tau_1(-\ell)$, $M'_{f_1}(\sigma')$, $M_{f_1}(\sigma)$: Points located on the lower side Σ_{f_1} of Σ_f

$B^\tau_2(-\ell)$, $M_{f_2}(\sigma')$, $M_{f_2}(\sigma)$: Points located on the upper side Σ_{f_2} of Σ_f

Figure 12 - Small Motion of the First Kind. Growth of the Vortex Sheet Behind a Deeply Submerged Body

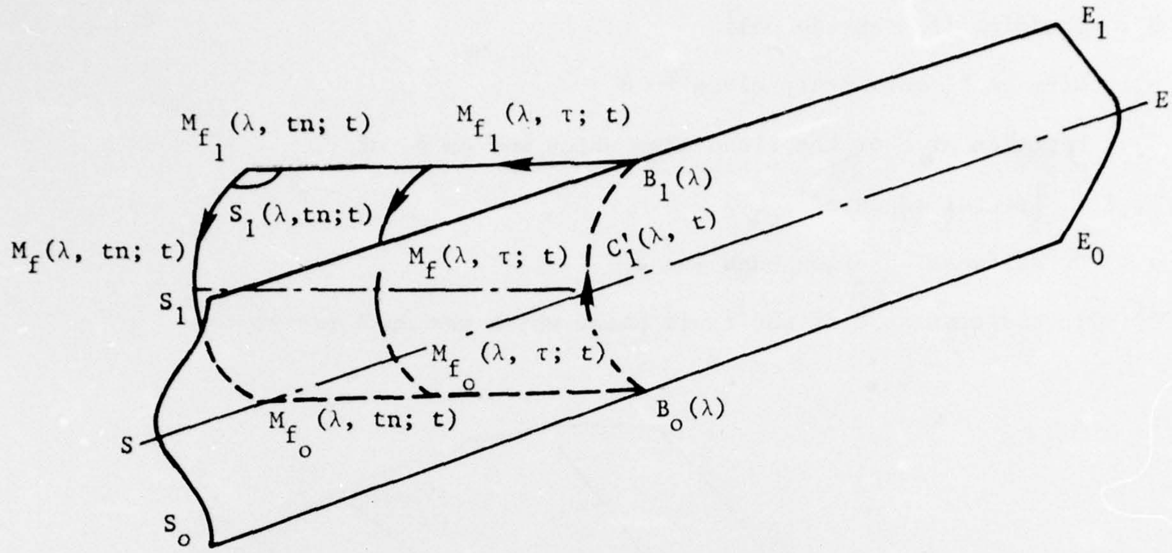


Figure 13 - Double Model in Swaying Motion at $t \varepsilon (t_n, t_{n+1})$ $S_1(\lambda, t_n; t)$ is the Surface Generated by the Free Vortex Arc Shed from $(B_o(\lambda), B_1(-\lambda)$ during the Time Interval (t_n, t)

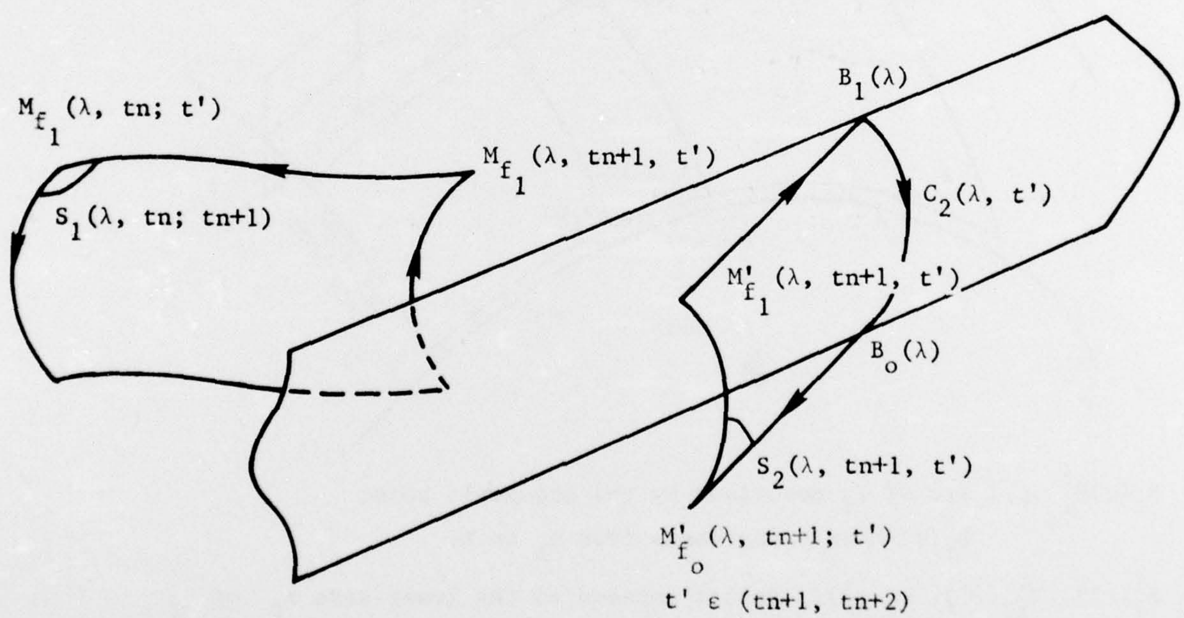


Figure 14 - Double Model in Swaying Motion at $t\varepsilon (t_{n+1}, t_{n+2})$ The Surfaces $S_1(\ , t_n, t_n + {}_1)$ are now Quite Free. $S_2(\ , t_n + {}_1; t)$ *is Generated by the Free Vortex Arc Shed to Starboard from* $B_o(\ ), B_1(- \ )$ during the Time Interval $(t_n + {}_1, t)$

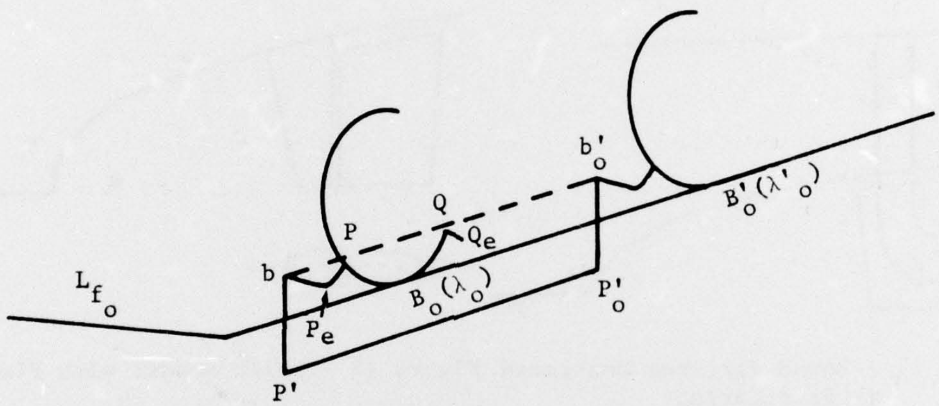


Figure 15 - Geometry for Derivation of Generalized Kutta Condition in the Case of Small Motions of the Second Kind

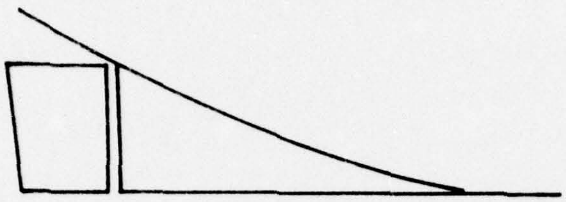


Figure 16 a - Rudder Behind a Skeg

Figure 16 b - Rudder Without a Skeg

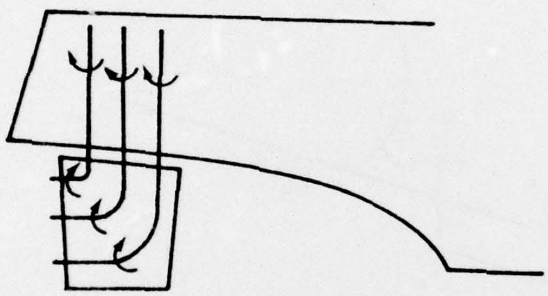


Figure 17 - Bound Vortices Continued to Free Surface

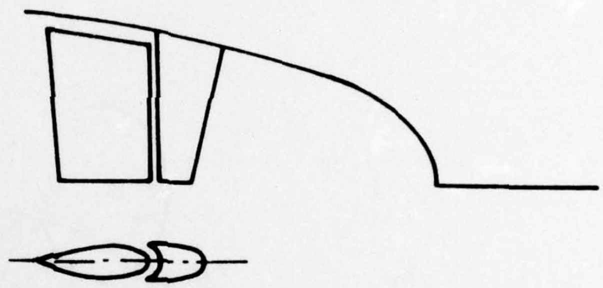


Figure 18 - Split Rudder with Fixed Strut

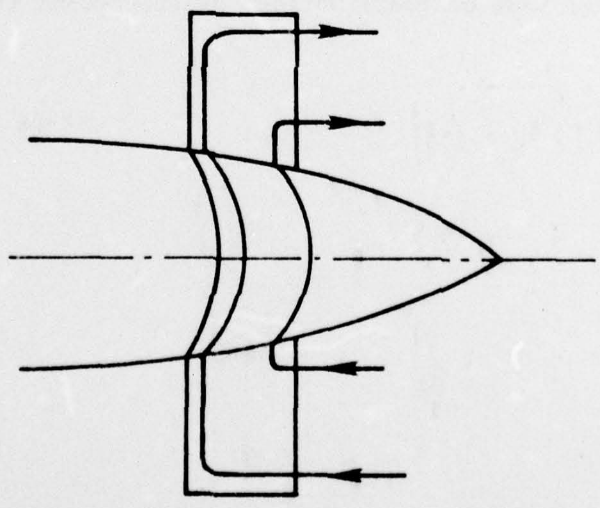


Figure 19 - Bound Vortices on Submarine

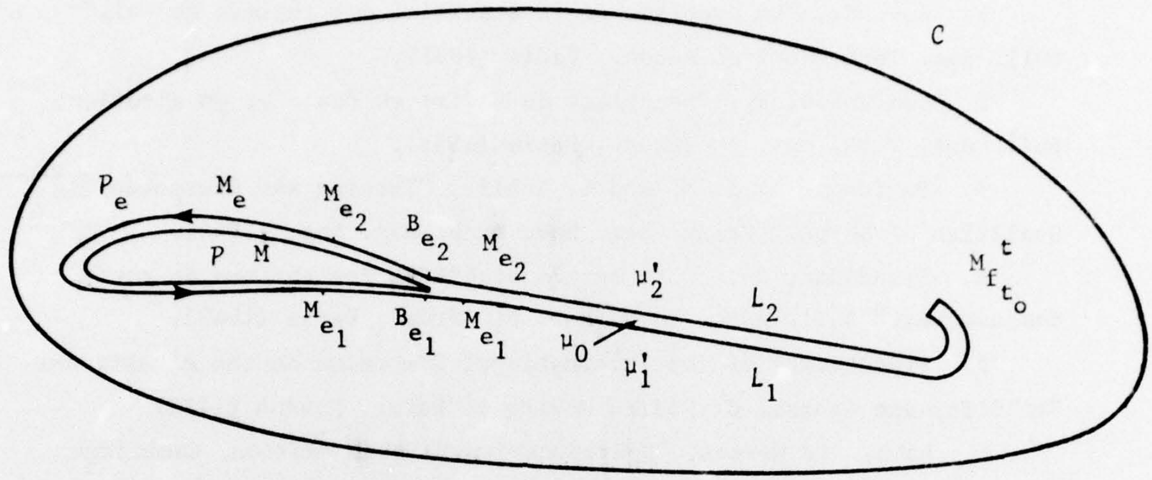


Figure 20 - Small Unsteady Motion of a Wing Profile

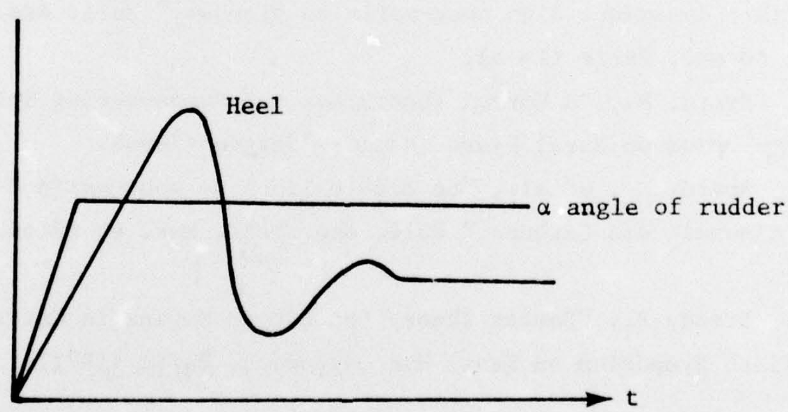


Figure 21 - Heeling Motion of a Submarine

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Page	Line	Write	Instead of
82	Formula (5.58)	$-k^2\cos^2\theta\hat{\Phi}'_2$	$-k^2\cos^2\theta\Phi'_2$
82	16	$d\Sigma_0(P_0)\}$	$d\Sigma_0(P_0)$
85	-11	curves	the curves
93	last line	$t_o = -\infty,\ B(\sigma')M_f(\sigma') = -c(t-\tau)$	$t_o = -\infty\ B(\sigma')M_f(\sigma') = -c(t-\tau)$
99	- 1	S_1S_1, S_0, E_1, E_0	$EE_0E_1S_0$, S_1 and S
101	13	of (t_n, t_{n+1})	of $t_n, t_{n+1})$
107	3	$\vec{V}^t_{2n}$	V^t_{2n}
109	Formula (6.47)	$\vec{ds}$	ds
111	- 8	center of gravity	CG
113	9	(O'x', O'y')	(O'x, O'y)
113	13	x'-axis	X'-axis
114	Formula (7.2)	$\Gamma(t) = -\int_{B_{e1}\ldots B_{e2}}$	$\Gamma(t) = \int_{B_{e1}\ldots B_2}$
114	- 5	$-\Phi(\mu'_1, t)$	$-\Phi(\mu', t)$
114-117	Various	all μ'_1, μ'_2, μ'_{e_1} and μ'_{e_2}	all μ_1, μ_2, μ_{e_1} and μ_{e_2}
115	- 3	add (7.6) at the end of the equation	

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117	11	$t=t_1+0$	$t=tz+0$
120-121	Various	all Ω	all $\acute{\omega}$
123	- 6	delete "been"	been
140	last line	$B_o(\ )B_1(\ )$	$B_o(\ ),\ B_1(-\ )$
143	Figure 20	delete M_{e_1} between B_{e_1} and μ_1'	
		delete M_{e_2} between B_{e_2} and μ_2'	